

LIST COLOURINGS OF MULTIPARTITE HYPERGRAPHS

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ABSTRACT. Let $\chi_l(G)$ denote the list chromatic number of the r -uniform hypergraph G . Extending a result of Alon for graphs, Saxton and the second author used the method of containers to prove that, if G is simple and d -regular, then $\chi_l(G) \geq (1/(r-1) + o(1)) \log_r d$.

To see how close this inequality is to best possible, we examine $\chi_l(G)$ when G is a random r -partite hypergraph with n vertices in each class. The value when $r = 2$ was determined by Alon and Krivelevich; here we show that $\chi_l(G) = (g(r, \alpha) + o(1)) \log_r d$ almost surely, where d is the expected average degree of G and $\alpha = \log_n d$.

The function $g(r, \alpha)$ is defined in terms of “preference orders” and can be determined fairly explicitly. This is enough to show that the container method gives an optimal lower bound on $\chi_l(G)$ for $r = 2$ and $r = 3$, but, perhaps surprisingly, apparently not for $r \geq 4$.

1. INTRODUCTION

Let G be an r -uniform hypergraph: that is to say, its edges are sets of r vertices. For brevity, we often call G an r -graph: thus a 2-graph is just a graph. Given an assignment $L : V(G) \rightarrow \mathcal{P}(\mathbb{N})$ of a list $L(v)$ of colours to each vertex v , we say G is L -chooseable if, for each vertex v , it is possible to choose a colour $c(v) \in L(v)$, such that there is no edge e with $c(v)$ the same for all $v \in e$. The minimum number k such that G is L -choosable whenever $|L(v)| \geq k$ for every v , is called the *list-chromatic* number of G , denoted by $\chi_l(G)$. This notion was introduced for graphs by Vizing [29] and by Erdős, Rubin and Taylor [9]. In [9] it was proved, amongst other things, that $\chi_l(K_{d,d}) = (1 + o(1)) \log_2 d$, and also that the determination of $\chi_l(K_{d,d})$ is intimately related to the study of “Property B” (namely, the study of the minimum number of edges in a non-bipartite uniform hypergraph). The $o(1)$ term here, as elsewhere in this paper, denotes a quantity tending to zero as $d \rightarrow \infty$. (This is a convenient place to point out that logarithms to various different bases appear in this paper but, where no base is specified, the logarithm is natural.)

The theorem of Erdős, Rubin and Taylor was extended by Alon and Krivelevich [2], who proved that $\chi_l(G) = (1 + o(1)) \log_2 d$ holds almost surely for a random bipartite graph with n vertices in each class, edges being present independently with probability p , provided $d = pn > d_0$ for some constant d_0 . (They actually proved something a little sharper, and they showed that this more general result is also tied to Property B.)

Alon [1] proved that *every* graph G of average degree d satisfies $\chi_l(G) \geq (1/2 + o(1)) \log_2 d$. The value $1/2$ can, in fact, be replaced by 1 here (see below), and so it follows that complete bipartite graphs, and more generally random bipartite graphs, are graphs whose list chromatic number is (more or less) minimal amongst graphs of given average degree.

When $r \geq 3$ it is not true, in general, that the list chromatic number of an r -graph grows with its average degree. For example, if F is a 2-graph and G is

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an r -graph on the same vertex set, such that every edge of G contains an edge of F , then $\chi_l(G) \leq \chi_l(F)$, but the average degree of G can be large, even if that of F is not and $\chi_l(F)$ is small. However, examples of this kind can be avoided by considering *simple* r -graphs, in which different edges have at most one vertex in common. For this reason, we are particularly interested in simple hypergraphs.

The case when the edges of G form a Steiner triple system was studied by Haxell and Pei [11], who proved $\chi_l(G) = \Omega(\log d / \log \log d)$. Haxell and Verstraëte [12] obtained the bound $\chi_l(G) \geq (1 + o(1)) (\log d / 5 \log \log d)^{1/2}$ for every simple d -regular 3-graph, and Alon and Kostochka [3] showed that $\chi_l(G) \geq (\log d)^{1/(r-1)}$ for every simple r -graph G of average degree d ; in particular $\chi_l(G)$ grows with d .

The correct rate of growth was found by Saxton and Thomason [24], who proved, via the container method, that $\chi_l(G) \geq \Omega(\log d)$ if G is a simple d -regular r -graph (see [26] for a refinement of the argument). An improved bound was obtained in [25], with an extension to not-quite-simple r -graphs in [27]: we state it here.

Proposition 1.1 ([25, 27]). *Let $r \in \mathbb{N}$ be fixed. Let G be an r -graph with average degree d . Suppose that, for $2 \leq j \leq r$, each set of j vertices lies in at most $d^{(r-j)/(r-1)+o(1)}$ edges, where $o(1) \rightarrow 0$ as $d \rightarrow \infty$. Then*

$$\chi_l(G) \geq (1 + o(1)) \frac{1}{(r-1)^2} \log_r d.$$

Moreover, if G is regular then

$$\chi_l(G) \geq (1 + o(1)) \frac{1}{r-1} \log_r d.$$

In particular, for 2-graphs, the $1/2$ in Alon's bound can be replaced by 1, which is tight, as described above. Thus the container method gives a best possible bound for graphs. Does it also give an optimal bound for r -graphs (at least simple r -graphs) when $r \geq 3$? That is the question underlying the results of this paper.

To answer this question, it is natural, in the light of what is known about 2-graphs, to examine r -partite r -graphs. (Every r -graph contains an r -partite subgraph whose average degree is less by only a constant factor, so a lower bound on χ_l for r -partite graphs applies to all r -graphs. On the other hand, non- r -partite d -regular simple r -graphs can have χ_l as large as $\Omega(d / \log d)$, so for upper bound purposes we consider only r -partite r -graphs.) Our r -graphs G will have order rn , with vertex set $V = V_1 \cup V_2 \cup \dots \cup V_r$, the V_i 's being disjoint sets of size n . Each edge of G has exactly one vertex in each V_i .

1.1. Properties of r -partite r -graphs. A simple random argument, mimicking Erdős's work on Property B [7], shows that if G is such an r -partite r -graph then $\chi_l(G) \leq \log_r n + 2$. (Suppose $|L(v)| = \ell$ for all v . Throughout the paper we use the word *palette* for the set $\bigcup_{v \in V(G)} L(v)$ (or a superset of it); it is a set containing all colours in all lists. For each colour in the palette, select some V_i at random, and forbid the colour to be chosen by any vertex in V_i . Then the expected number of vertices v having every colour in $L(v)$ forbidden is $rn r^{-\ell}$. So if $rn r^{-\ell} < 1$ then G is L -chooseable.) This bound holds even for complete r -partite r -graphs — that is, where every possible edge is present. If $r \geq 3$, these r -graphs are not simple. But it is not difficult to construct a simple d -regular r -partite r -graph G with n not much larger than d , thereby giving examples of simple d -regular r -graphs with $\chi_l(G) \leq (1 + o(1)) \log_r d$.

It follows from these remarks and from Proposition 1.1 that the minimum value of $\chi_l(G)$ amongst simple d -regular r -graphs lies between $(1/(r-1) + o(1)) \log_r d$ and $(1 + o(1)) \log_r d$. In the light of the case $r = 2$, one might expect the minimum

to be attained by r -partite r -graphs of order rn with n close to d , or by random r -partite graphs, and so these are the objects we study.

An important definition is the following. Given an r -partite r -graph as just described, and a subset $X \subset V$, let $X_i = X \cap V_i$. We define i_X to be the index i such that $|X_i|$ is largest. For the sake of definiteness, if there is more than one such index we take i_X to be the smallest, though any choice would do. Thus we define

$$i_X = \min\{i : |X_i| = \max\{|X_j| : 1 \leq j \leq r\}\}.$$

We can now define the two properties of G that will matter to us. Both properties involve a condition on sets X stated in terms of the product of all $|X_i|$ except the largest, that is, a product of $r - 1$ quantities. The first condition is about independent sets, meaning sets X that contain no edge of G . The second is about degenerate sets: as usual, we say that X is k -degenerate if, for every non-empty $Y \subset X$, the subgraph $G[Y]$ has a vertex of degree at most k . Degenerate sets are relevant here because they are easily coloured, as noted in Lemma 3.2.

Definition 1.2. Let G be an r -uniform r -partite hypergraph as just described. Let d be a real number with $1 \leq d \leq n^{r-1}$.

- G has *property* $I(r, n, d)$ if every independent set X satisfies

$$\prod_{i \neq i_X} |X_i| < \frac{n^{r-1}}{d} \log^2 d. \quad (1)$$

- G has *property* $D(r, n, d)$ if every set X that satisfies

$$\prod_{i \neq i_X} |X_i| < \frac{n^{r-1}}{d} \quad (2)$$

is $4(\log d / \log \log d)$ -degenerate.

The properties are useful when the parameter d is equal, or near to, the average degree, though it is convenient not to make this a requirement. In particular, note that if G has property $D(r, n, d)$ then it has $D(r, n, d')$ for every $d' > d$; indeed every G has property $D(r, n, n^{r-1})$ since the only set then satisfying (2) is $X = \emptyset$. Similarly, if G has property $I(r, n, d)$ then it has $I(r, n, d')$ for $e^2 < d' < d$. The interesting values of d are those for which G has both $D(r, n, d)$ and $I(r, n, d)$. The apparently strange dependence on d in the definitions is not crucial to our main theorem: we need only an expression close to n^{r-1}/d in each of (1) and (2). The definitions are stated in the way they are in order to comply, rather loosely, with properties of random hypergraphs when d is the expected average degree.

Theorem 1.3. *There is a number $d_0 = d_0(r)$ such that the following holds. Let $G \in \mathcal{G}(n, r, p)$ be a random r -partite r -uniform hypergraph where $p = p(n)$, and let $d = pn^{r-1} \geq d_0$. Then G almost surely has properties $I(r, n, d)$ and $D(r, n, d)$.*

This theorem is entirely routine: the point of it is that it gives examples of r -graphs having both properties. As we shall explain shortly, $\chi_l(G)$ can be determined very precisely for any r -graph G having both properties, and we can then compare this value with the lower bound given by Proposition 1.1.

The lower bound in Proposition 1.1 is better for regular G . Random r -graphs are close to regular but not quite regular. In fact the lower bound given for regular r -graphs holds for such close-to-regular graphs too, but it is worth noting the existence of regular r -graphs having the two properties.

Theorem 1.4. *There is a number $d_0 = d_0(r)$ such that the following holds. Let d be an integer with $d \geq d_0$ and let $n \geq r^5 d^4$. Then there is a simple d -regular r -partite r -graph G having properties $I(r, n, d)$ and $D(r, n, d)$.*

1.2. List chromatic numbers. Our main result is that the list chromatic number of hypergraphs satisfying both properties can be determined more or less exactly. It will be expressed in terms of the function $g(r, \alpha)$, a function defined via what we call *preference orders*. The precise definition is delayed to §2 because it needs a little discussion.

The parameter α , however, can be explained now: it will always be true that $\alpha = \log_n d$, where d is as in Definition 1.2. We specified $d \geq 1$ in that definition so that α is well-defined. Since $1 \leq d \leq n^{r-1}$ we always have $0 \leq \alpha \leq r - 1$. Notice that, if G is simple, then the average degree is at most n , and so (since we are imagining d in the definition to be the average degree), simple hypergraphs are associated with the range $0 \leq \alpha \leq 1$. Similarly, complete r -partite hypergraphs are associated with $\alpha = r - 1$.

Here, at last, is the main theorem.

Theorem 1.5. *Let G be an r -uniform r -partite hypergraph that satisfies the properties $I(r, n, d)$ and $D(r, n, d)$ of Definition 1.2. Then*

$$\chi_l(G) = (g(r, \alpha) + o(1)) \log_r d.$$

Here, $\alpha = \log_n d$, the function $g(r, \alpha)$ is described in terms of preference orders by Definition 2.5, and the $o(1)$ term tends to zero as $d \rightarrow \infty$.

The theorem is a bit opaque without any information about the function $g(r, \alpha)$, so we describe some of its properties immediately. It is, in fact, quite a straightforward function: in particular, for $r = 2$ and $r = 3$ it is constant, and for every r it is constant over the range $0 \leq \alpha \leq 1$ associated with simple hypergraphs. Moreover in §7 we give an explicit formula for what we believe is the exact value of $g(r, \alpha)$, though we have no proof.

Theorem 1.6. *For each $r \in \mathbb{N}$, $r \geq 2$, the function $g(r, \alpha)$ maps $[0, r - 1]$ to $[0, 1]$ as follows:*

- (a) $g(r, \alpha)$ is continuous and decreasing (that is, non-increasing) in α ,
- (b) $g(r, r - 1) = 1/(r - 1)$,
- (c) $g(2, \alpha) = 1$ for $0 \leq \alpha \leq 1$ and $g(3, \alpha) = 1/2$ for $0 \leq \alpha \leq 2$,
- (d) for $r \geq 4$, $g(r, \alpha)$ is constant for $0 \leq \alpha \leq 1 + 1/(r + 3)$,
- (e) $g(4, 0) = 0.3807\dots$, and
- (f) $g(r, 0) \sim (\log r)/r$ as $r \rightarrow \infty$.

The fact that $g(2, \alpha) = 1$ means the case $r = 2$ of Theorem 1.5 is the theorem of Alon and Krivelevich [2], though without an explicit bound on the error term.

Likewise, the fact that $g(r, r - 1) = 1/(r - 1)$ means that, if G is complete (and we take $d = n^{r-1}$) then $\chi_l(G) = (1/(r - 1) + o(1)) \log_r d = (1 + o(1)) \log_r n$, as noted at the outset of §1.1.

As mentioned earlier, our motivation is to investigate whether the lower bound on χ_l supplied by Proposition 1.1 for regular simple r -graphs, namely $(1/(r - 1) + o(1)) \log_r d$, is tight. We also suggested that, amongst all simple regular r -graphs, “random-like” r -partite ones would likely have lowest list-chromatic number. In the light of Theorem 1.3, most such r -graphs enjoy properties $I(r, n, d)$ and $D(r, n, d)$, so their list-chromatic number is given by Theorem 1.5. Now $0 \leq \alpha \leq 1$ for simple r -graphs, and $g(r, \alpha)$ is constant in this range, so the question of whether the above approach shows Proposition 1.1 to be tight now comes down to the question of whether $g(r, 0) = 1/(r - 1)$.

As can be seen from Theorem 1.6, $g(r, 0) = 1/(r - 1)$ indeed holds for $r = 2$ and $r = 3$, and so Proposition 1.1 is tight in these cases. For $r \geq 4$ we have been unable to determine the exact value of $g(r, 0)$, but we can prove that $g(r, 0) > 1/(r - 1)$. Hence the bound in Proposition 1.1 appears not to be tight — indeed, we think it

more likely that the lower bound $(g(r, 0) + o(1)) \log_r d$ might hold in general for all simple r -graphs of average degree d .

It turns out that the reason why a gap emerges between Theorem 1.4 and Proposition 1.1 only for $r \geq 4$ is that, for $r = 2$, preference orders are more or less trivial, and even for $r = 3$ optimal preference orders are tightly constrained. It is only when $r \geq 4$ that there is room for more interesting preference orders to exist; more detail appears in §7.

As mentioned, we think that $(g(r, 0) + o(1)) \log_r d$ might be a lower bound on $\chi_l(G)$ for every r -uniform simple hypergraph G of average degree d , and to prove this it would be enough to do it for r -partite graphs. In order to obtain a lower bound it is necessary to show that there is a list function $L : V(G) \rightarrow \mathcal{P}(\mathbb{N})$ with $|L(v)| = (g(r, 0) + o(1)) \log_r d$ for all $v \in V(G)$, such that G is not L -chooseable. In practice the best lists for this job appear to be random lists, such as in the proof of Theorem 4.4, where the bound is proved for r -graphs having property $D(r, n, d)$. We don't have such a proof for all r -graphs, but we can prove a complementary result, namely, that for any d -regular r -partite r -graph G , if random lists of size larger than $g(r, 0) \log_r d$ are assigned, then G is L -chooseable. (It is necessary to impose a weak bound on n in terms of d for the usual reason that, if we make too many random choices, bad things are bound to happen.)

Theorem 1.7. *Let $\epsilon > 0$ and $M > 1$ be given. Let G be a simple d -regular r -partite r -uniform hypergraph with $n \leq d^M$ vertices in each class. For each $v \in V(G)$ let a list $L(v)$ of size $\ell = \lfloor (1 + \epsilon)g(r, 0) \log_r d \rfloor$ be chosen uniformly at random from a palette of size $t \geq \ell$, independently of other choices. Then, with probability tending to one as $d \rightarrow \infty$, G is L -chooseable.*

It is somewhat curious, to us at least, that preference orders are used in the proof of Theorem 1.5 in two entirely different ways, both in the upper bound (obtained from a colouring algorithm designed around preference orders — this is how we first came across them), and also in the lower bound (for a different reason). This “coincidence” is reminiscent of the relationship with Property B in the graph case.

As stated earlier, we define preference orders in §2 and discuss them enough to be able to define the function $g(r, \alpha)$. Then, in §3 we describe the colouring algorithm and prove Theorems 1.7 and 3.3; the latter theorem is one half of Theorem 1.5, giving an upper bound for $\chi_l(G)$ when G has property $D(r, n, d)$. A corresponding lower bound, for graphs with property $I(r, n, d)$, is given by Theorem 4.4 in §4, and this provides the other half of Theorem 1.5. The elementary probabilistic argument behind Theorem 1.3 is given in §5, and the twist needed for Theorem 1.4 follows in §6. Then, in §7, we examine preference orders in more detail, and describe how to calculate, or at least to estimate, the function $g(r, \alpha)$; we put some effort into this since it is, of course, at the heart of the paper. Finally in §8 we comment briefly on the relationship between preference orders and Property B.

We use standard notation for intervals of real numbers, such as $[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, and we denote by $[n]$ the set of integers $\{1, 2, \dots, n\}$.

2. PREFERENCE ORDERS

In this section we introduce the notion of preference orders, and define $g(r, \alpha)$.

To motivate the ideas, consider the most basic case of our problem, where G is a simple d -regular 3-uniform 3-graph with d vertices in each class (that is, $n=d$): such a graph is precisely the graph of a Latin square. As mentioned in §1.1, $\chi_l(G) \leq \log_3 d + 2$, but this bound holds as well for complete 3-partite 3-graphs. For a lower bound, we have $\chi_l(G) \geq (1/2 + o(1)) \log_3 d$ from Proposition 1.1. The upper bound comes from forbidding each colour on one of the vertex classes, chosen at random

for each colour. To improve the bound we must allow some colours to appear in *every* class: we call these colours *free* and the other colours *forbidden*. Suppose, for each colour, we make it free with probability $1 - 3q$ and otherwise forbid it on one of V_1 , V_2 and V_3 , with probability q each. A vertex $v \in V_i$ now chooses a non-free colour from $L(v)$ if possible (meaning a colour forbidden on some V_j , $j \neq i$), but if there are no such, it chooses a free colour. Once again, v has no available choice if every colour in $L(v)$ is forbidden on V_i , and we want the expected number of such vertices to be small, say $3dq^\ell < 1/2$. But there is now another potential problem, which is the presence of monochromatic edges; if each vertex of an edge chooses a free colour (for each vertex this happens with probability $(1 - 3q)^\ell$) then the colours chosen might be the same. The expected number of edges where each vertex chooses a free colour is at most $d^2(1 - 3q)^\ell$ (we must allow for the lists to be overlapping) so we require $d^2(1 - 3q)^\ell < 1/2$. Taking say $q = 3/10$ and $\ell = 0.92 \log_3 d + 2$ makes both expectations small; hence $\chi_l(G) \leq 0.92 \log_3 d + 2$.

To get a further improvement, we look for a strategy which will reduce the likelihood of each vertex in an edge picking the same free colour. For each of V_1 , V_2 and V_3 , decide an order of preference on the palette $\bigcup_{v \in V(G)} L(v)$: denote these orderings by $<_1$, $<_2$ and $<_3$. The triple $P = (<_1, <_2, <_3)$ is called a *preference order*. Then the choice of $c(v) \in L(v)$ is made as follows: if $v \in V_i$, let $c(v)$ be a non-free colour in $L(v)$ if one is available, else let $c(v)$ be the most preferred free colour according to the order $<_i$. We should design the orderings $<_1$, $<_2$ and $<_3$ so that a colour preferred in one class is deprecated in another. A good way to do this is in example P_c below. In this manner the likelihood of a monochromatic edge is reduced and, in fact, using the preference order P_c we obtain $\chi_l(G) \leq 0.78 \log_3 d + 3$, as verified in Theorem 3.1; this is the best bound we have for Latin square graphs in general, but the algorithm works only for graphs with a small number of vertices.

We can make further progress if we know something of the structure of G . We cannot demand that every set of a certain size is independent, but we can hope to describe sparse sets, and that is what property $D(r, n, d)$ is doing. For our algorithm to make use of these sparse sets, we modify it slightly so that v does not commit immediately to the most preferred free colour in $L(v)$ but, rather, v promises to restrict its choice to within some small named subset of similarly preferred colours in $L(v)$. If P is well designed then the collection of vertices promising to use the same subset spans a sparse subgraph, and the colouring can then be completed (details are in §3).

What is a good design of preference order P ? We assign a value to each P (Definition 2.3), and pick the P of best value: this value is specifically designed so that the number of vertices choosing a given colour ties up with the kind of sparse sets guaranteed by property $D(r, n, d)$.

Are there other ways to use a preference order in a colouring algorithm? In the simplest conceivable algorithm, each vertex just commits at once to the most preferred colour in its list. Perhaps surprisingly, such an algorithm is weak (giving no improvement over $\log_3 d$). To make a gain we need either to use forbidden colours, as we do in Theorem 3.1, or to incorporate the method of restrictive promises, as we do elsewhere, using the algorithm set out in detail in §3. This algorithm makes no use of forbidden colours; it turns out these give no extra benefit when restrictive promises are used.

In summary, a preference order is, more or less, a specification of r orders of preference on the palette, one order for each V_i . If the orderings are all the same then the same colours will be preferred in each class and a proper colouring is unlikely to be achieved. When $r = 2$, and G is a bipartite graph, then, intuitively, one would expect the best palette order for V_2 to be the reverse of that on V_1 ,

and indeed this is the case — in fact this method reproduces known results about Property B (see §8). What constitutes a good preference order for $r \geq 3$ is what we shall study and, as hinted at before, whereas it is easy to answer the question for $r = 3$, the answer for $r \geq 4$ is surprisingly elusive.

Let us get down to specifics.

Definition 2.1. Let $<$ be a total ordering of the set $[m]$. Given $k \in [m]$, the *relative position* $\text{rpos}_<(k)$ of k in the ordering is $1/m$ times the number of elements less than or equal to k . So $\text{rpos}_< : [m] \rightarrow \{1/m, 2/m, \dots, m/m\}$ is a bijection and

$$\text{rpos}_<^{-1}(1/m) < \text{rpos}_<^{-1}(2/m) < \dots < \text{rpos}_<^{-1}(1).$$

Definition 2.2. An (r, m) -*preference order* is an r -tuple $P = (<_1, \dots, <_r)$ where $<_i$ is a total ordering of $[m]$, $1 \leq i \leq r$. Abusing notation, we write $x \in P$ if $x \in (0, 1]^r$ and there is some $k \in [m]$ such that $x = (\text{rpos}_{<_1}(k), \dots, \text{rpos}_{<_r}(k))$.

Thus $x \in P$ means x is the tuple of relative positions of some element of $[m]$. Notice that $\{x : x \in P\}$ determines P to within a permutation of $[m]$, because each $x \in P$ tells us the relative position in each order of some element $k \in [m]$, but we do not know which element. Since the actual labels of the elements in the ground set $[m]$ are usually unimportant (for example, when using P in the algorithm above we generally begin by randomly mapping the palette to $[m]$), we often think of the set $\{x : x \in P\}$ as specifying P .

Here are three examples of preference orders. The *identity* ordering is the ordering $1 < 2 < 3 < \dots < m$.

P_a Let $r = 2$, let $<_1$ be the identity ordering, and let $<_2$ be the reverse of $<_1$; that is, $m <_2 (m-1) <_2 \dots <_2 1$. Then

$$\{x : x \in P_a\} = \{(k/m, 1 + 1/m - k/m) : k \in [m]\}.$$

P_b Let $r = 3$ and let $m = 3p$ be a multiple of three. Let $<_1$ be the identity ordering and let $<_2, <_3$ be “rotations” of $<_1$ by p and by $2p$ elements, meaning that

$$\begin{aligned} &2p+1 <_2 \dots <_2 3p <_2 1 <_2 \dots <_2 p <_2 p+1 <_2 \dots <_2 2p \\ &p+1 <_3 \dots <_3 2p <_3 2p+1 <_3 \dots <_3 3p <_3 1 <_3 \dots <_3 p. \end{aligned}$$

Then

$$\begin{aligned} \{x : x \in P_b\} &= \{(i/m, 1/3 + i/m, 2/3 + i/m) : i \in [p]\} \\ &\cup \{(1/3 + i/m, 2/3 + i/m, i/m) : i \in [p]\} \\ &\cup \{(2/3 + i/m, i/m, 1/3 + i/m) : i \in [p]\}. \end{aligned}$$

P_c This is the same as P_b except that, in each of $<_1, <_2$ and $<_3$ we reverse the order of bottom third of the elements, that is, we reverse the order of those elements with relative positions $1/m$ to p/m . So

$$\begin{aligned} &p <_1 \dots <_1 1 <_1 p+1 <_1 \dots <_1 2p <_1 2p+1 <_1 \dots <_1 3p \\ &3p <_2 \dots <_2 2p+1 <_2 1 <_2 \dots <_2 p <_2 p+1 <_2 \dots <_2 2p \\ &2p <_3 \dots <_3 p+1 <_3 2p+1 <_3 \dots <_3 3p <_3 1 <_3 \dots <_3 p, \end{aligned}$$

and

$$\begin{aligned} \{x : x \in P_c\} &= \{(1/3 + 1/m - i/m, 1/3 + i/m, 2/3 + i/m) : i \in [p]\} \\ &\cup \{(1/3 + i/m, 2/3 + i/m, 1/3 + 1/m - i/m) : i \in [p]\} \\ &\cup \{(2/3 + i/m, 1/3 + 1/m - i/m, 1/3 + i/m) : i \in [p]\}. \end{aligned}$$

It turns out that P_c is an essentially optimal choice of preference order when $r = 3$. The next definition defines a parameter of a preference order, designed to measure its effectiveness in our colouring algorithm. The parameter captures the way the algorithm makes use of various independent sets. The form of the definition reflects the properties of sparse sets in the r -graphs we are interested in, set out in properties $I(r, n, d)$ and $D(r, n, d)$. This is explained in a little more detail just before Theorem 3.3.

Analogously to the definition of i_X for a set $X \subset V(G)$, we define, for an r -tuple $x = (x_1, x_2, \dots, x_r) \in [0, 1]^r$

$$i_x = \min\{i : x_i = \max\{x_j : 1 \leq j \leq r\}\},$$

that is, i_x is a specific index of a largest x_j .

Definition 2.3. Let P be an (r, m) -preference order. Let $0 \leq \theta \leq 1/r$. Then

$$f_P(\theta) = \max \left\{ \prod_{i \neq i_x} x_i : x \in [\theta, 1]^r, x \in P \right\}.$$

Observe that $f_P(\theta)$ depends only on $\{x : x \in P\}$, supporting the earlier remark that it is this set that matters rather than P itself. Observe too that the set in the definition is non-empty, because there are fewer than m/r numbers $k \in [m]$ with $\text{rpos}_{<_1}(k) < 1/r$, and likewise for $<_2, \dots, <_r$, so there is some k with $\text{rpos}_{<_i}(k) \geq 1/r \geq \theta$ for all i . That is, there is some $x \in [1/r, 1]^r$ with $x \in P$. In particular, $f_P(\theta) \geq (1/r)^{r-1}$.

Notice that, by definition, $f_P(\theta)$ is non-increasing in θ . The value we are mostly interested in is $f_P(0)$, the maximum of $\prod_{i \neq i_x} x_i$ over all $x \in [0, 1]^r$. This value, when $\theta = 0$, relates to the case $0 \leq \alpha \leq 1$ in Theorem 1.5. The reader who wishes, from now on, to consider only $\theta = 0$ will not miss out on anything of substance.

It is necessary to allow larger θ in order to handle larger α . Somewhat vaguely, this is because as α increases to $r - 1$, meaning d increases to n^{r-1} , then the range narrows of those $x \in P$ that play an interesting role, and θ captures this reduced range. For more, we refer to the proofs of Theorems 3.3 and 4.4.

Consider the three examples P_a , P_b and P_c above. For $x \in P_a$ we have $x = (x_1, x_2) = (k/m, 1 + 1/m - k/m)$ for some $k \in [m]$. Then $x_1 + x_2 = 1 + 1/m$, so one of x_1, x_2 is at most $1/2 + 1/2m$ and the other is at least $1/2 + 1/2m$. Thus i_x is the index of the larger co-ordinate and $\prod_{i \neq i_x} x_i = l/m$ for some $l \leq (m + 1)/2$. Therefore $f_{P_a}(0) = 1/2$ if m is even and $f_{P_a}(0) = 1/2 + 1/2m$ if m is odd. Moreover it can be seen that $f_{P_a}(\theta) = f_{P_a}(0)$ for $0 \leq \theta \leq 1/r = 1/2$, since the maximum value of $\prod_{i \neq i_x} x_i = l/m$ is always attained by some x with $x \in [1/2, 1]^2$.

For $x = (x_1, x_2, x_3) \in P_b$ it can be seen that one co-ordinate exceeds $2/3$ and the other two are i/m and $1/3 + i/m$ for some $i \leq p = m/3$. Thus $\prod_{i \neq i_x} x_i = (i/m)(1/3 + i/m)$ and $f_{P_b}(0) = 2/9$. The maximum is achieved by some x with $\min x_i \geq 1/3$ and so, once again, $f_{P_b}(\theta) = f_{P_b}(0)$ for $0 \leq \theta \leq 1/r = 1/3$.

For $x = (x_1, x_2, x_3) \in P_c$, one co-ordinate exceeds $2/3$ and the other two are $1/3 + 1/m - i/m$ and $1/3 + i/m$ for some $i \leq p = m/3$. Thus $\prod_{i \neq i_x} x_i = (1/3 + 1/m - i/m)(1/3 + i/m)$ and $f_{P_c}(0) = 1/9 + 1/3m$. The maximum is achieved by some x with $\min x_i \geq 1/3$ and so, once again, $f_{P_c}(\theta) = f_{P_c}(0)$ for $0 \leq \theta \leq 1/r = 1/3$.

In the three examples, $f_P(\theta)$ is constant for $0 \leq \theta \leq 1/r$. This is a reflection of the fact, noted in §1.2, that the overall situation is more straightforward for $r \leq 3$ and new phenomena appear only when $r \geq 4$.

It turns out that the best preference orders for the colouring algorithm are those with the lowest values of f_P . This leads us to the next definition.

Definition 2.4. Let $r \geq 2$. For $0 \leq \theta \leq 1/r$ we define

$$\begin{aligned} f(r, \theta, m) &= \min \{ f_P(\theta) : P \text{ is an } (r, m)\text{-preference order} \} \\ \text{and } f(r, \theta) &= \inf \{ f(r, \theta, m) : m \in \mathbb{N} \}. \end{aligned}$$

It was noted that $f_P(\theta)$ is non-increasing in θ , and hence so are $f(r, \theta, m)$ and $f(r, \theta)$. Moreover we saw that $f_P(\theta) \geq (1/r)^{r-1}$ for all P and θ , so $f(r, \theta) \geq (1/r)^{r-1}$ for all θ . The examples P_a and P_c show that $f(2, \theta) \leq 1/2$ for $0 \leq \theta \leq 1/2$ and $f(3, \theta) \leq 1/9$ for $0 \leq \theta \leq 1/3$. Hence equality holds in each of these cases. In particular, when $r = 2, 3$, then $f(r, \theta)$ is constant for $0 \leq \theta \leq 1/r$.

We are, at last, in a position to define $g(r, \alpha)$. To do this, we need to relate a value of θ to each α . Formally, this special value is $\beta(\alpha) = \sup \{ \theta : \theta^\alpha \leq f(r, \theta) \}$. However, we shall show later that $f(r, \theta)$ is continuous; since it is also decreasing, and θ^α is increasing, there is a unique solution to $\theta^\alpha = f(r, \theta)$, and this is $\beta(\alpha)$. So, anticipating continuity, we take this simpler statement as the definition.

Definition 2.5. Let $r \geq 2$ and $0 < \alpha \leq r - 1$. Define $\beta = \beta(\alpha)$ by $\beta^\alpha = f(r, \beta)$. Then we define $g(r, \alpha) = -1/\log_r(f(r, \beta))$. Note that if $\alpha = \log_n d$ then

$$f(r, \beta)^{g(r, \alpha) \log_r d} = \frac{1}{d} \quad \text{and} \quad \beta^{g(r, \alpha) \log_r d} = \frac{1}{n}. \quad (3)$$

Observe that $g(r, 0)$ is not defined by this statement but, since $g(r, \alpha)$ is constant for $0 < \alpha \leq 1$ (see Theorem 1.6) then we define $g(r, 0)$ to equal this constant value.

We remark that $\beta(r - 1) = 1/r$ because $f(r, 1/r) = (1/r)^{r-1}$ (see Theorem 2.6 below). Moreover $\beta(\alpha)$ is strictly increasing: for if $\alpha_1 < \alpha_2$ and $\beta(\alpha_1) \geq \beta(\alpha_2)$, then $f(r, \beta(\alpha_1)) = \beta(\alpha_1)^{\alpha_1} > \beta(\alpha_1)^{\alpha_2} \geq \beta(\alpha_2)^{\alpha_2} = f(r, \beta(\alpha_2))$, contradicting the fact that $f(r, \theta)$ is decreasing.

Notice how the expression $g(r, \alpha) \log_r d$, appearing in Theorem 1.5, appears also in (3). In the proof of the theorem, we try to appeal to (3) directly rather than to the definition of $g(r, \alpha)$.

The next theorem lists some basic properties of $f(r, \theta)$, in the same way that Theorem 1.6 lists some of those of $g(r, \alpha)$. In particular it shows that $f(r, \theta)$ is constant for small θ , which is the reason $g(r, \alpha)$ is constant for small α .

Theorem 2.6. For each $r \in \mathbb{N}$, $r \geq 2$, the function $f(r, \theta)$ maps $[0, 1/r]$ to $[0, 1]$ as follows:

- (a) $f(r, \theta)$ is continuous and decreasing in θ ,
- (b) $f(r, 1/r) = (1/r)^{r-1}$,
- (c) for $r > 2$, $f(r, \theta) \leq f(r - 1, \theta)$,
- (d) $f(2, \theta) = 1/2$ for $0 \leq \theta \leq 1/2$ and $f(3, \theta) = 1/2$ for $0 \leq \theta \leq 1/3$,
- (e) for $r \geq 4$, $f(r, \theta)$ is constant for $0 \leq \theta \leq (1 - 1/r)e^{-r+1}$,
- (f) $f(4, 0) = 0.0262\dots$, and
- (g) $((r - 1)/er)^{r-1} \leq f(r, 0) \leq (r - 1)!/r^{r-1}$.

Theorems 1.6 and 2.6 are proved in §7.

3. A LIST COLOURING ALGORITHM AND SOME UPPER BOUNDS

In order to prove $\chi_l(G) \leq \ell$ for some ℓ , we need an algorithm that will colour G whenever the vertices are given lists of ℓ colours each.

We start with a proof of an upper bound for Latin square graphs, mentioned earlier in §2, which uses preference orders in an elementary way. The proof makes no use of the structure of the graph and does not, in fact, require simplicity. It does make use of randomization.

Theorem 3.1. Let G be a d -regular 3-partite 3-graph with vertex classes of size d . Then $\chi_l(G) \leq 0.78 \log_3 d + 3$.

Proof. Let $\ell = \lceil 0.78 \log_3 d \rceil + 2$ and assume each vertex v has a list $L(v)$ of ℓ colours to choose from. Let m be the size of the palette $\bigcup_{v \in V(G)} L(v)$; by increasing m if need be, we can assume m is divisible by 3. Take a random map $\Phi : \bigcup_{v \in V(G)} L(v) \rightarrow [m]$, and let P_c be the $(3, m)$ -preference order given as an example in §2.

Let $q^2 = (1 - 2q)/9$, so $q = (-1 + \sqrt{10})/9 \approx 0.24$. As described in §2, each colour in the palette is forbidden on one of V_1, V_2 or V_3 , with probability q each, and is otherwise free, with probability $1 - 3q$. If $v \in V_i$ then $c(v)$ is taken to be a non-free colour, if $L(v)$ has one available, else it is the free colour whose image under Φ is most preferred in the ordering $<_i$.

There are two ways the colouring can fail: a vertex might have no colour available, or an edge might be monochromatic. The expected number of vertices with no colours available, that is, all colours in $L(v)$ are forbidden on V_i , is $3dq^\ell < 1/2$. Suppose now some edge $e = \{v_i, v_2, v_3\}$ is monochromatic, where $v_i \in V_i$, $1 \leq i \leq 3$: say $c(v_1) = c(v_2) = c(v_3) = \gamma$. Then γ must be free. Observe that any other colour lying in more than one of $L(v_1), L(v_2)$ and $L(v_3)$ is free, else it would have been chosen by one of the vertices. Let $\Phi(\gamma) = k$ and let $p_i = \text{rpos}_{<_i}(k)$, $1 \leq i \leq 3$; for ease of notation assume $k \leq m/3$ so $p_1 \leq p_2 \leq p_3$. If $\gamma' \in L(v_1)$ is free and $\gamma' \neq \gamma$ then $\Phi(\gamma') <_1 \Phi(\gamma)$. By the definition of P_c this means $\Phi(\gamma) <_2 \Phi(\gamma')$ and $\Phi(\gamma) <_3 \Phi(\gamma')$, so $\gamma' \notin L(v_2) \cup L(v_3)$. Thus $L(v_1) \cap L(v_i) = \{\gamma\}$ for $i = 2, 3$; let $j = |L(v_2) \cap L(v_3)|$.

Consider the event M_e that e is monochromatic (necessarily of colour γ , given what we now know of $L(v_i)$). Let $p'_i = p_i - 1/m$, $i = 1, 2$. The probability that $c(v_1) = \gamma$ is at most $(q + (1 - 3q)p'_1)^{\ell-1}$, because every colour in $L(v_1) \setminus \{\gamma\}$ must either be forbidden on V_1 or must map under Φ to a relative position below p_1 . Treating in like manner $(L(v_2) \cap L(v_3)) \setminus \{\gamma\}$, $L(v_2) \setminus L(v_3)$ and $L(v_3) \setminus L(v_2)$, we have

$$\Pr(M_e) \leq (q + (1 - 3q)p'_1)^{\ell-1} ((1 - 3q)p'_2)^{j-1} (q + (1 - 3q)p'_2)^{\ell-j} (q + (1 - 3q))^{\ell-j}.$$

Since $p'_2 \leq 2/3$ and $q \approx 1/4$ we have $q > (1 - 3q)p'_2$, so $(1 - 3q)p'_2 \leq (q + (1 - 3q)p'_2)/2 < (q + (1 - 3q)p'_2)(1 - 2q)$. Hence the bound for $\Pr(M_e)$ decreases with j , and so

$$\Pr(M_e) \leq [(q + (1 - 3q)p'_1)(q + (1 - 3q)p'_2)(1 - 2q)]^{\ell-1}.$$

But $p'_1 = 1/3 - x - 1/m \leq 1/3 - x$ and $p'_2 = 1/3 + x$ for some $x \geq 0$, so $(q + (1 - 3q)p'_1)(q + (1 - 3q)p'_2) \leq 1/9$. Thus $\Pr(M_e) \leq ((1 - 2q)/9)^{\ell-1} = q^{2\ell-2}$.

Finally, there are d^2 edges in G , so the expected number of monochromatic edges is at most $d^2 q^{2\ell-2} < 1/2$. Hence there is some mapping Φ for which every vertex has a choice of colour and for which no edge is monochromatic, proving the theorem. \square

As discussed in §2, the algorithm used in Theorem 3.1 is too weak for general use, and we turn now to the main algorithm. It too uses randomized preference orders.

Algorithm for list colouring an r -partite r -graph G having lists of size ℓ .

- Let $[t]$ be the palette. Choose parameters k and δ . Let $m = \delta\ell/k$.
- Randomly partition the palette into m blocks B_1, \dots, B_m of equal size (increase t if need be). Choose an (r, m) -preference order $P = (<_1, \dots, <_r)$.
- Let $\mathcal{B} = \{B_1, \dots, B_m\}$. Say $B \in \mathcal{B}$ is *available* to $v \in V(G)$ if $|L(v) \cap B| > k$.
- Define $b : V(G) \rightarrow \mathcal{B}$ by $b(v) = B_q$ where, if $v \in V_i$, then q is the member of $\{j : B_j \text{ is available to } v\}$ of greatest relative position in the order $<_i$.
- For $B \in \mathcal{B}$ let $X(B) = \{v : b(v) = B\}$. Colour $G[X(B)]$ using colours from B .

We shall choose $\delta < 1$ small. Since, for $v \in V(G)$, at most $mk = \delta\ell$ colours are in blocks unavailable to v , there are at least $(1-\delta)\ell$ colours in $L(v)$ in available blocks: in particular $b(v)$ is well-defined. In effect, v is promising to choose a colour $c(v)$ from the block $b(v)$, this block being the most preferred amongst blocks available to v (where $v \in V_i$ uses the order $<_i$). The algorithm will succeed — that is, it will show G is L -chooseable, if for each $B \in \mathcal{B}$ we can colour $G[X(B)]$ using colours from B , because the sets $X(B)$ partition $V(G)$ and the sets B partition $[t]$.

Since $|L(v) \cap B| > k$ for each $v \in X(B)$, the algorithm will succeed if the subgraph $G[X(B)]$ is k -degenerate, as verified by applying the next (standard and elementary) lemma to $H = G[X(B)]$.

Lemma 3.2. *Let H be a k -degenerate r -graph. Let $L : V(H) \rightarrow \mathcal{P}(\mathbb{N})$ be a list assignment with $|L(v)| > k$ for every vertex v . Then H is L -chooseable.*

Proof. Construct an ordering v_1, \dots, v_n of the vertices of H in which v_j has minimum degree in the subgraph $H[\{v_1, \dots, v_j\}]$, $1 \leq j \leq n$. Now, for $j = 1, \dots, n$ in turn, choose a colour $c(v_j) \in L(v_j)$ as follows. There are at most k edges in $H[\{v_1, \dots, v_j\}]$ that contain v_j : select a vertex other than v_j in each of these edges, and then choose $c(v_j) \in L(v_j)$ different from the colours of the selected vertices (possible since $|L(v_j)| > k$). The resultant colouring is a proper colouring of H . \square

We give two examples of the use of the algorithm. In each case, proving that the algorithm succeeds amounts to showing that $G[X(B)]$ is k -degenerate, for each block $B \in \mathcal{B}$. The first example supplies the upper bound for Theorem 1.5, the second example establishes Theorem 1.7.

Broadly speaking, the first example works for the following reason. There is an r -tuple $x = (x_1, \dots, x_r)$ of relative positions of the block B in the preference order P . If x_i is small then the number $|X_i|$ of vertices in V_i for which $b(v) = B$ will, very likely, be correspondingly small. If $x_j < \theta$ for some j (where θ is determined by α), it turns out that $X_j = \emptyset$, so certainly $G[X]$ is k -degenerate. On the other hand, if $x_j \geq \theta$ for all j , then by definition of $f_P(\theta)$ we know $\prod_{i \neq i_x} x_i \leq f_P(\theta)$. This leads to a bound on $\prod_{i \neq i_x} |X_i|$ which, because of property $D(r, n, d)$, again means $G[X]$ is k -degenerate. The second example works in a similar way but finishes differently; because $\prod_{i \neq i_x} |X_i|$ and X is a random set (as the lists were chosen randomly) we again conclude that $G[X]$ is k -degenerate.

We give a quantitative bound in the first example, with a rate at which the $o(1)$ term tends to zero as $d \rightarrow \infty$. This bound depends on two factors, one being the value of k for which the sets in property $D(r, n, d)$ are k -degenerate, and the other being the rate at which $f(r, \theta, m) \rightarrow f(r, \theta)$ as $m \rightarrow \infty$. It turns out to be the second of these that predominates in our analysis; we use a bound on the rate proved in §7.

Theorem 3.3. *Let $r \geq 2$. Then there exists $d_1 = d_1(r)$ such that, if $d > d_1$ and G is an r -uniform r -partite hypergraph with property $D(r, n, d)$, then*

$$\chi_l(G) \leq (g(r, \alpha) + (\log \log d)^{-1/5}) \log_r d$$

where $\alpha = \log_n d$.

Proof. All estimates in the proof hold provided $d_1(r)$ is large enough: we ignore integer parts. Let lists of ℓ colours be assigned to each vertex of G , where $\ell = (g(r, \alpha) + (\log \log d)^{-1/5}) \log_r d$. Let $[t]$ be the palette comprising all the colours in all the lists; clearly $t \geq \ell$. Define $k = 4 \log d / \log \log d$ and $\delta = (\log \log d)^{-1/4}$. Further define $m = \delta\ell/k$. By adding a few dummy colours to the palette if necessary, we may assume that t is divisible by m .

Let $\beta = \beta(\alpha)$ as specified in Definition 2.5. There is some (r, m) -preference order $P = (\prec_1, \dots, \prec_r)$ with $f_P(\beta) = f(r, \beta, m)$. Apply the algorithm above to G , using k , δ and P as just specified. What remains is to show that $G[X(B)]$ is k -degenerate for each $B \in \mathcal{B}$.

Here is the central part of the argument. Consider some particular block B , and let $X = X(B)$. Let $x = (x_1, \dots, x_r)$ be the r -tuple of relative positions of B in the preference order P : that is, if, say, $B = B_j$, then x_i is the relative position of j in \prec_i . Let $v \in X_i$. We know that at least $(1 - \delta)\ell$ of the colours in v 's list lie in available blocks, and, by definition of X , these blocks all lie in relative positions x_i or below in the i 'th order. There are $x_i t$ colours from $[t]$ in blocks B or below it in the i th order, so the probability that the random partition of $[t]$ into blocks results in $(1 - \delta)\ell$ of v 's colours being placed in these low blocks is at most

$$\binom{\ell}{(1 - \delta)\ell} \binom{x_i t}{(1 - \delta)\ell} \binom{t}{(1 - \delta)\ell}^{-1} \leq \binom{\ell}{\delta\ell} x_i^{(1 - \delta)\ell} \leq \left(\frac{e}{\delta}\right)^{\delta\ell} x_i^{(1 - \delta)\ell}.$$

Hence, by Markov's inequality, the inequality $|X_i| \leq rm(e/\delta)^{\delta\ell} x_i^{(1 - \delta)\ell} n$ holds with probability exceeding $1 - 1/rm$, and thus, with probability more than $1 - 1/m$, the inequality holds for $1 \leq i \leq r$. Consequently, with positive probability, there exists a partition of $[t]$ such that the inequality holds for every block B , and for every set $X_i = X(B) \cap V_i$, $1 \leq i \leq r$.

To finish the proof, it is now enough to check that if $x = (x_1, \dots, x_r) \in P$, and $X \subset V(G)$ satisfies $|X_i| \leq rm(e/\delta)^{\delta\ell} x_i^{(1 - \delta)\ell} n$, then X is k -degenerate. There are two possibilities: either $\prod_{i \neq i_x} x_i \leq f_P(\beta)$, or $\prod_{i \neq i_x} x_i > f_P(\beta)$.

Consider the first possibility, that $\prod_{i \neq i_x} x_i \leq f_P(\beta)$. Then

$$\prod_{i \neq i_x} |X_i| \leq (rm)^r \left(\frac{e}{\delta}\right)^{r\delta\ell} \left(\prod_{i \neq i_x} x_i\right)^{(1 - \delta)\ell} n^{r-1} \leq (rm)^r \left(\frac{e}{\delta}\right)^{r\delta\ell} f_P(\beta)^{(1 - \delta)\ell} n^{r-1}.$$

Now $f_P(\beta) = f(r, \beta, m) \leq f(r, \beta) + 2^r \sqrt{(\log rm)/m}$ by Lemma 7.8. Theorem 2.6 tells us that $f(r, \beta)$ and $g(r, \alpha)$ are bounded below (namely $f(r, \beta) \geq f(r, 1/r) = (1/r)^{r-1}$ and $g(r, \alpha) \geq g(r, 1/r) = 1/(r - 1)$) so, recalling the definitions of k , δ and m , we have $f_P(\beta) \leq f(r, \beta)(1 + \delta) \leq f(r, \beta)e^\delta$. Put $\Lambda = (\log_r d)(\log \log d)^{-1/5}$, so $\ell = g(r, \alpha) \log_r d + \Lambda$. Then, using (3), we obtain

$$\prod_{i \neq i_x} |X_i| \leq (rm)^r \left(\frac{e}{\delta}\right)^{r\delta\ell} f(r, \beta)^{(1 - \delta)\ell} e^{\delta\ell} n^{r-1} = (rm)^r \left(\frac{e}{\delta}\right)^{r\delta\ell} f(r, \beta)^{\Lambda - \delta\ell} e^{\delta\ell} \frac{n^{r-1}}{d}.$$

By Theorem 2.6, $f(r, \beta) \leq f(2, 0) = 1/2$, so we conclude that $\prod_{i \neq i_x} |X_i| \leq K n^{r-1}/d$, where $K = (rm)^r (e/\delta)^{r\delta\ell} e^{\delta\ell} 2^{-\Lambda + \delta\ell}$. Since Λ is much larger than either $\delta\ell \log(1/\delta)$ or $\log m$, we see that $K < 1$. Hence $\prod_{i \neq i_x} |X_i| < n^{r-1}/d$ and, because G has property $D(r, n, d)$, this means $X = X(B)$ is k -degenerate, so resolving the first of the two possibilities.

Consider now the second possibility, where $\prod_{i \neq i_x} x_i > f_P(\beta)$. By definition of $f_P(\beta)$ there must be some index j with $x_j < \beta$. Therefore, using equation (3), and the fact that (by definition) $\beta(\alpha) \in [0, 1/r]$, we have

$$|X_j| < rm(e/\delta)^{\delta\ell} \beta^{(1 - \delta)\ell} n = rm(e/\delta)^{\delta\ell} \beta^{\Lambda - \delta\ell} \leq rm(e/\delta)^{\delta\ell} r^{-\Lambda + \delta\ell} < K,$$

where Λ and K are as in the previous paragraph. But we saw that $K < 1$, and so $|X_j| < 1$, meaning $X_j = \emptyset$. But then X contains no edges, and so is certainly k -degenerate. This resolves the second of the two possibilities, completing the proof of the theorem. \square

Our second example of the use of the algorithm is a proof of Theorem 1.7.

Proof of Theorem 1.7. Let G and the lists $L(v)$ be as stated. We choose constants k and δ as follows. First, write $\ell = g(r, 0) \log_r d + \Lambda$, so $\Lambda \approx \epsilon g(r, 0) \log_r d$. Then choose $\delta < 1$ small enough that $(r\ell)^r (e/\delta)^{r\delta\ell} e^{\delta\ell} 2^{-\Lambda+\delta\ell} < 2^{-\Lambda/2}$ (assuming, as we may, that ℓ is large). Then choose k so that $2^{-k\Lambda/2} < 1/d^{M+1}$.

As usual, put $m = \delta\ell/k$ and assume t is a multiple of m . Let $\alpha = \log_n d$. Since G is simple, $d \leq n$; thus $\alpha \leq 1$ and (by Theorem 1.6) $g(r, \alpha) = g(r, 0)$. Select an (r, m) -preference order $P = (\prec_1, \dots, \prec_r)$ with $f_P(0) = f(r, 0, m)$. Apply the algorithm with k, δ and P as specified: we need only show that $G[X(B)]$ is k -degenerate for each block of colours $B \in \mathcal{B}$. Fix some block $B = B_j$ and, as in the proof of Theorem 3.3, let $x = (x_1, \dots, x_r) \in P$ be the tuple of relative positions of j in the orders \prec_1, \dots, \prec_r .

The vertex lists are chosen randomly. We can imagine the algorithm first makes the random partition of the palette, and afterwards the assignment of lists is made to the vertices. The first step determines the collections \mathcal{L}_i , $1 \leq i \leq r$, of vertex lists such that, if $v \in V_i$ and $L(v) \in \mathcal{L}_i$, then $b(v) = B$. The second step determines which vertices $v \in V_i$ receive a list from \mathcal{L}_i , namely, it determines X_i . Hence we can consider X_i to have been generated in the following way: first, its size $|X_i|$ is chosen from a binomial distribution with parameters $n, |\mathcal{L}_i|/\binom{t}{\ell}$, and then, having decided the size $|X_i|$, X_i itself is a random $|X_i|$ -subset of V_i . In fact, having partitioned the palette, we may choose the sizes $|X_i|$ for every $B \in \mathcal{B}$ and every i , $1 \leq i \leq r$, before choosing the sets X_i themselves. In the proof of Theorem 3.3, we showed if $v \in V_i$ and v has some list $L(v)$ then the probability that $b(v) = B$ is at most $(e/\delta)^{\delta\ell} x_i^{(1-\delta)\ell}$. But this probability is the probability that $L(v) \in \mathcal{L}_i$, and this equals $|\mathcal{L}_i|/\binom{t}{\ell}$; hence $|\mathcal{L}_i|/\binom{t}{\ell} \leq (e/\delta)^{\delta\ell} x_i^{(1-\delta)\ell}$. Using Markov's inequality again as in the proof of Theorem 3.3, we may assume that all the chosen sizes $|X_i|$ satisfy $|X_i| \leq rm(e/\delta)^{\delta\ell} x_i^{(1-\delta)\ell} n$.

We now re-use a calculation performed in the first possibility in the proof of Theorem 3.3, though much less care is needed with the estimates this time. Taking $\beta = 0$, and noting that $m = \Theta(\log d)$, we have once again $f_P(0) \leq f(r, 0)e^\delta$, and so $\prod_{i \neq i_X} (|X_i|/n) \leq K/d$, where $K = (rm)^r (e/\delta)^{r\delta\ell} e^{\delta\ell} 2^{-\Lambda+\delta\ell}$. Since $m < \ell$, we have $K < 2^{-\Lambda/2}$ by choice of δ .

Let $v \in V_{i_X}$ and let E be one of the $\binom{d}{k}$ choices of a set of k edges containing v . Given that G is simple, the probability, conditional on $v \in X$, that the edges in E lie within X is $\prod_{i \neq i_X} \binom{n-k}{|X_i|-k} / \binom{n}{|X_i|} \leq \prod_{i \neq i_X} (|X_i|/n)^k \leq (K/d)^k$. Thus the probability that the degree of v in $G[X]$ exceeds k is at most $\binom{d}{k} (K/d)^k \leq K^k < 2^{-k\Lambda/2} < 1/d^{M+1}$, by choice of k . This probability is less than $1/nd$, so with probability exceeding $1 - 1/d$, every vertex in X_{i_X} has degree at most k in $G[X]$; because G is r -partite this certainly implies $G[X]$ is k -degenerate.

So, given $B \in \mathcal{B}$, $G[X(B)]$ is k -degenerate with probability more than $1 - 1/d$, and since $|\mathcal{B}| = m = o(d)$ this means that, with probability tending to one, $G[X(B)]$ is k -degenerate for every $B \in \mathcal{B}$ and thus G is L -colourable, proving the theorem. \square

4. A LOWER BOUND

To prove the lower bound in Theorem 1.5 we shall choose some lists for G at random. We make use of the following basic tail estimate.

Proposition 4.1 ([15, Theorem 2.1, Theorem 2.8]). *If Y is binomially distributed, with mean λ , then $\mathbb{P}(Y \leq \lambda - y) \leq e^{-y^2/2\lambda}$. The same bound holds for any sum Y of independent Bernoulli variables.*

Remark 4.2. The bound of Proposition 4.1 holds if Y is hypergeometrically distributed. This can be proved either by a comparison of moment generating

functions, on which the inequality is based (Hoeffding [13]) or by showing that in this case Y is in fact a sum of independent Bernoulli variables (Vatutin and Mikhailov [28] — the proof is reproduced in [14] and the idea goes back at least to Harper [10]). More generally, the bound holds for variables of the form $Y = |X \cap T_1 \cdots \cap T_r|$, where $X, T_1, \dots, T_r \subset [n]$, X is fixed and T_1, \dots, T_r are chosen independently and uniformly of fixed sizes $|T_i| = t_i$, $1 \leq i \leq r$. When $r = 1$ then Y is hypergeometrically distributed: the general case can be derived from the generating function proof by induction on r , but in fact it is already shown explicitly in [28, Corollary 5] that Y of this form are sums of independent Bernoulli variables. The authors thank Svante Janson for pointing them to [28].

The next straightforward lemma provides the properties that we need of the lists. The size of the palette $[t]$ from which the lists are chosen is not particularly significant.

Lemma 4.3. *Let $\ell, n \in \mathbb{N}$, $\ell \geq 3$, and let $\zeta \in (0, 1]$. Let $t = \lceil 2\ell^2/\zeta \rceil$. Suppose that $n\zeta^\ell \geq 16t$. Then there exists a sequence $\mathcal{L} = (L_i)_{i \in [n]}$ of elements of $[t]^{(\ell)}$ such that, for every $Z \subset [t]$ with $|Z| = zt \geq \zeta t$, we have $|\{i \in [n] : L_i \subset Z\}| \geq n\zeta^\ell/4$.*

Proof. For each $i \in [n]$, choose L_i uniformly at random in $[t]^{(\ell)}$, independently of other choices. Let $Z \subset [t]$ have size $zt \geq \zeta t$. Let $Y = \{i \in [n] : L_i \subset Z\}$. Then Y is binomially distributed with parameters $n, p = \binom{|Z|}{\ell} / \binom{t}{\ell}$. By Proposition 4.1, $\Pr(Y \leq np/2) \leq \exp(-np/8)$. Now $np = n \binom{zt}{\ell} / \binom{t}{\ell} = n\zeta^\ell \prod_{i=0}^{\ell-1} (t - i/z) / (t - i) \geq n\zeta^\ell (1 - \ell/(z(t - \ell)))^\ell \geq n\zeta^\ell (1 - 1/(2\ell - 1))^\ell \geq n\zeta^\ell/2$, the penultimate inequality following from the fact that $t \geq 2\ell^2/z$ and the last because $\ell \geq 3$. Thus $\Pr(|Y| \leq n\zeta^\ell/4) \leq \Pr(Y \leq np/2) \leq \exp(-n\zeta^\ell/16) \leq \exp(-t)$. There are at most 2^t sets Z , so with positive probability $|\{i \in [n] : L_i \subset Z\}| \geq n\zeta^\ell/4$ holds for every $Z \subset [t]$, proving the lemma. \square

The next theorem establishes the lower bound in Theorem 1.5. The argument is roughly this. We assign lists of colours to the vertices using Lemma 4.3. Suppose it is possible to colour the graph. We obtain a preference order on the palette by letting $<_i$ be the order of popularity of the colours on V_i in this colouring. Thus there is some colour (green, say) whose relative positions $x = (x_1, \dots, x_r)$ satisfy $x_j > \theta$ for all j (θ determined by α) and $\prod_{i \neq i_x} x_i \geq f(r, \theta)$. By the properties of the lists this yields a lower bound on $\prod_{i \neq i_x} |X_i|$, where X is the set of vertices choosing green. But this lower bound is incompatible with G having property $I(r, n, d)$ and the fact that X is independent.

Theorem 4.4. *Let $r \geq 2$. Then there exists $d_2 = d_2(r)$ such that, if $d > d_2$ and G is an r -uniform r -partite hypergraph with property $I(r, n, d)$, then*

$$\chi_l(G) > g(r, \alpha) \log_r d - 6r \log \log d.$$

where $\alpha = \log_n d$.

Proof. All estimates hold provided d_2 is large enough: we ignore integer parts. Let G be a graph as in the theorem. Let $\ell = g(r, \alpha) \log_r d - 6r \log \log d$. Let $\zeta = \max\{\beta(\alpha), (1/r)^{r-1}\}$. Recall that $\beta(\alpha) \in [0, 1/r]$, and so $(1/r)^{r-1} \leq \zeta \leq 1/r$. Using (3), we have

$$n\zeta^{\ell+1} \geq n\beta(\alpha)^{g(r, \alpha) \log_r d} (1/r)^{-6r \log \log d + 1} = r^{6r \log \log d - 1}.$$

Thus $n\zeta^{\ell+1} \geq 2^{6r \log \log d - 1} \geq 2^{2 \log_2 \log_2 d + 6} = 64(\log_2 d)^2 \geq 64\ell^2$, since $g(r, \alpha) \leq 1$. Hence $n\zeta^\ell \geq 16t$ where $t = \lceil 2\ell^2/\zeta \rceil$. So we can apply Lemma 4.3 to obtain lists L_1, \dots, L_n of ℓ colours each. Assign these lists to the vertices in V_i , for each i , $1 \leq i \leq r$.

We claim that there is no vertex colouring compatible with these lists, and hence $\chi_l(G) > \ell$. Suppose, to the contrary, that there is such a colouring. Form an (r, t) -preference order, where the i th order on $[t]$ is determined by how frequently the colours are used on V_i . That is, in the i th order, the member of $[t]$ in relative position 1 is the colour appearing most often on V_i and the member in relative position $1/t$ is the colour appearing least often (ties can be broken arbitrarily). By Definition 2.4, there is some colour, green say, such that if x_i is the position of green in the i th order, then $x_i \geq \beta(\alpha)$ for $1 \leq i \leq r$, and $\prod_{i \neq i_x} x_i \geq f(r, \beta(\alpha), t) \geq f(r, \beta(\alpha))$. Because the second condition implies $x_i \geq f(r, \beta(\alpha))$ for $1 \leq i \leq r$, and Theorem 2.6 states $f(r, \beta(\alpha)) \geq f(r, 1/r) = (1/r)^{r-1}$, we have $x_i \geq \zeta$ for $1 \leq i \leq r$.

Let X be the set of vertices that are coloured green. We can find a lower bound for $|X_i|$ as follows. Let Z be the set of colours at or below relative position x_i in the i th order, that is, Z contains green and the colours less popular on V_i . Let X_i^* be the set of vertices in V_i that are coloured with some colour in Z . By definition of the i th order, $|X_i| \geq |X_i^*|/|Z| \geq |X_i^*|/t$.

Now $|Z| = x_i t$ because green has relative position x_i , and we know $x_i \geq \zeta$. So, by Lemma 4.3, at least $nx_i^\ell/4$ lists lie within Z , meaning at least $nx_i^\ell/4$ vertices in V_i have lists within Z . All of these vertices necessarily choose a colour in Z , and so lie within X_i^* . Therefore $|X_i^*| \geq nx_i^\ell/4$ and hence $|X_i| \geq |X_i^*|/t \geq nx_i^\ell/4t$.

Consequently, using equation (3), and writing f for $f(r, \beta(\alpha))$, noting that $f \leq f(2, 0) = 1/2$ (see Theorem 2.6), we have

$$\begin{aligned} \prod_{i \neq i_X} |X_i| &\geq \prod_{i \neq i_x} \frac{nx_i^\ell}{4t} \geq f^\ell \left(\frac{n}{4t} \right)^{r-1} = f^{g(r, \alpha) \log_r d} f^{-6r \log \log d} \left(\frac{n}{4t} \right)^{r-1} \\ &\geq \frac{n^{r-1}}{d} \left(\frac{2^{6 \log \log d}}{4t} \right)^{r-1}. \end{aligned}$$

Now $2^{6 \log \log d}/4t \geq \zeta 2^{6 \log \log d}/9\ell^2 \geq (1/r)^{r-1} 2^{6 \log \log d}/10 \log_r^2 d \geq \log^2 d$. Thus $\prod_{i \neq i_X} |X_i| \geq n^{r-1}(\log^2 d)/d$. But G has property $I(r, n, d)$ and so X cannot be an independent set, in contradiction to it being the set of green vertices in a proper colouring. \square

We remark that, if the set X in this proof were a random set of vertices, then the proof would work for every r -partite r -graph G even without assuming $I(r, n, d)$, because a random set with the specified lower bounds on $|X_i|$ would not be independent. In fact the set of vertices whose lists lie within Z is random, but there seems no reason why the set X itself should be random.

5. RANDOM r -PARTITE HYPERGRAPHS

We begin with a lemma that we shall use several times when treating various kinds of random r -partite hypergraphs on the vertex set $V_1 \cup \dots \cup V_r$.

Lemma 5.1. *Let some probability distribution be given on the space of subsets of $V = V_1 \cup \dots \cup V_r$. Let E be some event. Suppose, for each non-empty $X \subset V$, that $\Pr(X \in E) \leq (|X_{i_X}|/2en)^{(r+1)|X_{i_X}|}$ holds. Then $E = \emptyset$ almost surely, as $n \rightarrow \infty$.*

Proof. There are at most $(q+1)^r \leq 2^{rq}$ possibilities for the tuple $(|X_1|, \dots, |X_r|)$ if $|X_{i_X}| = q$, and, for each such possibility, the number of possible sets X is

$$\prod_i \binom{n}{|X_i|} \leq \left(\frac{en}{|X_{i_X}|} \right)^{|X_{i_X}|} \leq \left(\frac{en}{q} \right)^{rq},$$

because $(e/x)^x$ is an increasing function of x for $x \leq 1$. Hence the total probability that there is some set $X \in E$ is at most $\sum_{q \geq 1} 2^{rq} (en/q)^{rq} (q/2en)^{(r+1)q} =$

$\sum_{q \geq 1} (q/2en)^q$. Since $(x/2e)^x$ decreases for $0 < x \leq 1$, the first \sqrt{n} terms of this sum add to at most $\sqrt{n}(1/2en)$, which tends to zero. Since $q \leq n$, the remaining terms add to at most $\sum_{q \geq \sqrt{n}} (1/2e)^q$, which also tends to zero. Therefore E is almost surely empty. \square

The proof of Theorem 1.3 involves a routine verification. In fact, we do slightly more work than we need to, though the extra effort involved is negligible. We show that G almost surely has the two stronger properties $I'(r, n, d)$ and $D'(r, n, d)$. Property $I'(r, n, d)$ asserts that every set X containing at most $n/2d^{1/(r-1)}$ edges satisfies (1), and Property $D'(r, n, d)$ asserts that every set X satisfying (2) is $(4(\log d / \log \log d) - 1)$ -degenerate. The reason for adding this complication is that we can copy over the proof directly for use again in §6.

Proof of Theorem 1.3. Let $G \in \mathcal{G}(n, r, p)$ be a random r -partite r -uniform hypergraph and let $d = pn^{r-1} \geq d_0$.

Let $X \subset V(G)$ and let $x_i = |X_i|/n$. Let S be the number of edges in $G[X]$. Then $S \in \text{Bi}(\prod_{i=1}^r |X_i|, p)$, having mean $\lambda = p \prod_{i=1}^r |X_i| = d |X_{i_X}| \prod_{i \neq i_X} x_i$.

Let E be the collection of sets $X \subset V(G)$ such that $\prod_{i \neq i_X} |X_i| \geq n^{r-1}(\log^2 d)/d$ but $G[X]$ has at most $n/2d^{1/(r-1)}$ edges. To show that G almost surely has property $I'(r, n, d)$, we must show $E = \emptyset$ almost surely, and to do this we apply Lemma 5.1. Let $X \subset V$. If $\prod_{i \neq i_X} |X_i| < n^{r-1}(\log^2 d)/d$ then $X \notin E$ so $\Pr(X \in E) = 0$. If $\prod_{i \neq i_X} |X_i| \geq n^{r-1}(\log^2 d)/d$ then $\Pr(X \in E)$ is the probability that $S \leq n/2d^{1/(r-1)}$. In this case, $\prod_{i \neq i_X} x_i \geq (\log^2 d)/d$ so $\lambda > |X_{i_X}| \log^2 d$. Moreover $|X_{i_X}| \geq (\prod_{i \neq i_X} |X_i|)^{1/(r-1)} > n/d^{1/(r-1)}$. Hence certainly $\Pr(X \in E) \leq \Pr(S \leq \lambda/2) \leq e^{-\lambda/8}$ by Proposition 4.1, so $\Pr(X \in E) \leq e^{-|X_{i_X}|(\log^2 d)/8}$. Therefore, for Lemma 5.1 to apply, it is enough to show that $e^{-(\log^2 d)/8} \leq (|X_{i_X}|/2en)^{r+1}$. But $|X_{i_X}| \geq n/d^{1/(r-1)}$ so we need only show that $e^{-(\log^2 d)/8} \leq (1/2ed^{1/(r-1)})^{r+1}$, which easily holds if d is large.

Now let $\mathcal{X} = \{X \subset V : \prod_{i \neq i_X} |X_i| \leq n^{r-1}/d\}$. To show that G almost surely has property $D'(r, n, d)$ we must show, almost surely, that every $X \in \mathcal{X}$ is $(k-1)$ -degenerate, where $k = 4 \log d / \log \log d$. Notice that if $X \in \mathcal{X}$ and $Y \subset X$ then $Y \in \mathcal{X}$, and therefore to show every $X \in \mathcal{X}$ is $(k-1)$ -degenerate it suffices to show that every $X \in \mathcal{X}$ is either empty or has a vertex of degree at most $k-1$. We shall in fact show that if $X \in \mathcal{X}$ and $X \neq \emptyset$ then $G[X]$ contains fewer than $k|X_{i_X}|$ edges, and so the largest class of X has a vertex of degree less than k .

So let $E = \{X \in \mathcal{X} : X \neq \emptyset, S \geq k|X_{i_X}|\}$, where S is the number of edges in $G[X]$. We wish to show that $E = \emptyset$ almost surely, and we again use Lemma 5.1. Since $S \in \text{Bi}(\prod_{i=1}^r |X_i|, p)$, the probability that $S \geq k|X_{i_X}|$ is at most

$$\left(\frac{\prod_i |X_i|}{k|X_{i_X}|} \right) p^{k|X_{i_X}|} \leq \left(\frac{ep \prod_i |X_i|}{k|X_{i_X}|} \right)^{k|X_{i_X}|} = \left(\frac{ed \prod_{i \neq i_X} x_i}{k} \right)^{k|X_{i_X}|},$$

where $x_i = |X_i|/n$. To apply the lemma successfully, we need $(ed \prod_{i \neq i_X} x_i/k)^k \leq (|X_{i_X}|/2en)^{r+1}$, or $s \leq 1$ where $s = (2en/|X_{i_X}|)^{r+1} (ed \prod_{i \neq i_X} x_i/k)^k$. Let $z = d^{-1/(r-1)}$. For $|X_{i_X}| \leq zn$, we use the inequality $\prod_{i \neq i_X} x_i \leq (|X_{i_X}|/n)^{r-1}$, and so $s \leq (2en/|X_{i_X}|)^{r+1} (ed(|X_{i_X}|/n)^{r-1}/k)^k$: this is an increasing function of $|X_{i_X}|$ (we can assume $k > 3$ because d_0 is large) and so $s \leq (2e/z)^{r+1} (edz^{r-1}/k)^k = (2e/z)^{r+1} (e/k)^k$. For $|X_{i_X}| \geq zn$, we use instead that $\prod_{i \neq i_X} x_i \leq 1/d$ because $X \in \mathcal{X}$, and therefore $s \leq (2en/|X_{i_X}|)^{r+1} (e/k)^k \leq (2e/z)^{r+1} (e/k)^k$. Consequently $s \leq (2e/z)^{r+1} (e/k)^k \leq (2e)^{r+1} d^3 (e/k)^k$ holds for every $X \in \mathcal{X}$, and this bound is less than one because $k = 4 \log d / \log \log d$ and d_0 is large. This shows that, almost

surely, no $X \in \mathcal{X}$ has more than $k|X_{i_X}|$ edges, and almost surely G has property $D(r, n, d)$. \square

6. REGULAR r -PARTITE HYPERGRAPHS

In this section we aim to prove Theorem 1.4. Rather than apply the configuration model, which would work only for n much larger than d^4 , we work instead with the space $\mathcal{H}(n, r, d)$ of d -regular r -partite hypergraphs that are the union of d independently chosen 1-factors M_1, \dots, M_d . So M_i is a set of n pairwise disjoint edges, and M_1, \dots, M_d are chosen uniformly and independently from all possible 1-factors. (It is possible that hypergraphs in $\mathcal{H}(n, r, d)$ have multiple edges.)

An r -graph $H \in \mathcal{H}(n, r, d)$ is unlikely to be simple, but a small modification of it, \hat{H} , will be simple. Theorem 1.4 holds if \hat{H} has properties $I(r, n, d)$ and $D(r, n, d)$; for this to happen, we require H to satisfy $I'(r, n, d)$ and $D'(r, n, d)$, described in §5.

Lemma 6.1. *With probability tending to one as $d \rightarrow \infty$, $H \in \mathcal{H}(n, r, d)$ has properties $I'(r, n, d)$ and $D'(r, n, d)$.*

Proof. Let $H \in \mathcal{H}(n, r, d)$ be a random d -regular r -partite r -uniform hypergraph. Let $X \subset V(H)$ and let $x_i = |X_i|/n$. Let R be the number of edges in $H[X]$. Recall that in the proof of Theorem 1.3 we studied the distribution of a variable very similar to R , namely S , the number of edges in $G[X]$ where $G \in \mathcal{G}(n, r, p)$ and $d = pn^{r-1}$. Thus $\mathbb{E}S = p \prod_{i=1}^r |X_i| = d|X_{i_X}| \prod_{i \neq i_X} x_i$. When proving that G had property $I'(r, n, d)$ we used only that $\mathbb{E}S = d|X_{i_X}| \prod_{i \neq i_X} x_i$ and that the bound in Proposition 4.1 holds for S . We shall show that the same bound holds for R , and moreover $\mathbb{E}R = \mathbb{E}S$. Therefore the proof that G has $I'(r, n, d)$ can be used verbatim to show that H has $I'(r, n, d)$.

Let Z be the random variable that is the number of edges of M_1 lying inside X . For notational convenience, suppose $X_{i_X} = X_1$. Clearly $\mathbb{E}Z = |X_1| \prod_{i=2}^r x_i$, since the edge containing $v \in V_1$ has probability $\prod_{i=2}^r x_i$ of meeting each X_i , $i \geq 2$. Now M_1 can be generated from $r-1$ independent random bijections $V_i \rightarrow V_1$, $2 \leq i \leq r$, the edge of M_1 containing $v \in V_1$ being v together with those vertices that map to v . So $Z = |X_1 \cap T_2 \cap \dots \cap T_r|$, where T_i is the image of X_i , $2 \leq i \leq r$. By Remark 4.2, Z is a sum of independent Bernoulli variables. Finally, R is the sum of d independent copies of Z , so it too is a sum of independent Bernoulli variables, and hence Proposition 4.1 holds for R . Moreover $\mathbb{E}R = d\mathbb{E}Z = \mathbb{E}S$, and this completes the proof that H has $I'(r, n, d)$.

For the proof that H has $D'(r, n, d)$ we again copy from the proof of Theorem 1.3, and again assume $X_{i_X} = X_1$. Let $T \subset X_1$, $|T| = k_1$. The probability that $T \subset T_i$, where T_i is as in the previous paragraph, is $\binom{n-k_1}{|X_i|-k_1} \binom{n}{|X_i|}^{-1} \leq x_i^{k_1}$. Thus the probability is at most $(\prod_{i=2}^r x_i)^{k_1}$ that, for every $v \in T$, the edge of M_1 meeting v lies inside X . So the probability that X contains at least k_1 edges of M_1 is at most $\binom{|X_1|}{k_1} (\prod_{i=2}^r x_i)^{k_1}$. If $R \geq k|X_1|$, that is, $H[X]$ has at least $k|X_1|$ edges, then there are numbers k_1, \dots, k_d with $k_1 + \dots + k_d = k|X_1|$ such that X has k_j edges of M_j , $1 \leq j \leq d$. Thus $\Pr(R \geq k|X_1|) \leq \sum_{k_1 + \dots + k_d = k|X_1|} \prod_{j=1}^d \binom{|X_1|}{k_j} (\prod_{i=2}^r x_i)^{k_j} = \binom{d|X_1|}{k|X_1|} (\prod_{i=2}^r x_i)^{k|X_1|} \leq ((ed/k) \prod_{i=2}^r x_i)^{k|X_1|}$. But this is exactly the same as the bound on $\Pr(S \geq k|X_{i_X}|)$ that was used in the proof of Theorem 1.3, so, copying the rest of the proof verbatim, we have that H has $D'(r, n, d)$ almost surely. \square

The next lemma describes the modification of $H \in \mathcal{H}(n, r, d)$ that produces \hat{H} . Because H is close to simple, we can remove just a few edges to achieve simplicity, and replace them with well-chosen new edges to preserve regularity.

Lemma 6.2. *There is a number $d_3 = d_3(r)$ such that the following holds. Let d be an integer with $d \geq d_3$ and let $n \geq r^5 d^4$. Then, with probability at least $1/8$, $H \in \mathcal{H}(n, r, d)$ has the following property. There is a set I of at most $r^3 d^2$ independent (that is, pairwise disjoint) edges in H , and a set I' of $|I|$ independent edges none of which is in H , such that $H - I + I'$ is d -regular and simple.*

Proof. A pair of edges $\{e, f\}$ with $|e \cap f| \geq 2$ is called a *butterfly*. The *body* of the butterfly is $e \cap f$. The edges e and f are the *wings* of the butterfly. An r -graph is simple if it has no butterflies. We make a series of assertions, each of which holds with probability (conditional on previous assertions) at least $7/8$, if d_3 is large enough.

- (i) Every butterfly $\{e, f\}$ satisfies $|e \cap f| = 2$. This is because the expected number of butterflies with $|e \cap f| \geq 3$ is at most $\binom{r}{3} n^3 \binom{d}{2} (1/n^2)^2 < 1/8$. (To see this, let $\{u, v, w\} \subset e \cap f$. There are $\binom{r}{3}$ ways to choose classes V_i for u, v, w , n^3 ways to choose $\{u, v, w\}$ in the classes, and $\binom{d}{2}$ ways to choose matchings M_i containing e and M_j containing f . The probability that the edge of M_i containing u also contains $\{v, w\}$ is $1/n^2$, and likewise for M_j . Similar considerations explain subsequent assertions.)
- (ii) No two butterflies have the same body. This is because, assuming (i), the expected number of pairs of butterflies $\{e, f\}$ and $\{e, g\}$ with $e \cap f = e \cap g$ is at most $\binom{r}{2} n^2 \binom{d}{3} (1/n)^3 < 1/8$.
- (iii) Distinct butterflies have disjoint bodies. For suppose butterflies $\{e, f\}$ and $\{g, h\}$ have bodies $\{u, v\}$ and $\{u, w\}$, where $v \neq w$ by (ii). The expected number of such with $e = g$ is at most $r \binom{r}{2} n^3 d \binom{d}{2} (1/n^2) (1/n)^2 < 1/16$, and with $e \neq g$ is at most $r \binom{r}{2} n^3 3 \binom{d}{4} (1/n)^4 < 1/16$.
- (iv) No two butterflies share a wing. This is because, assuming (ii) and (iii), the expected number of pairs of butterflies $\{e, f\}$ and $\{e, g\}$ is at most $3 \binom{r}{4} n^4 d \binom{d}{2} (1/n^3) (1/n) (1/(n-1)) < 1/8$. Here the factors $1/n$ and $1/(n-1)$ arise from f and g containing their bodies, allowing for the fact that, conceivably, f and g come from the same matching M_i .
- (v) Distinct butterflies are disjoint. For suppose butterflies $\{e, f\}$ and $\{g, h\}$ have $e \cap g \neq \emptyset$. By (iv) we cannot have $|e \cap g| \geq 2$, for if $e = g$ then $\{e, f\}$ and $\{g, h\}$ share a wing, and if $e \neq g$ then $\{e, g\}$ is also a butterfly sharing a wing with both $\{e, f\}$ and $\{g, h\}$, which are distinct. Hence $|e \cap g| = \{u\}$ for some vertex u . By (iii) the expected number of these is at most $r \binom{r}{2}^2 n^5 \binom{d}{2}^2 (1/n^2)^2 (1/(n-1))^2 < 1/8$. Here we chose u and the two bodies, followed by e and g and by f and h .

Let b be the number of butterflies in H . The expected value of b is at most $\binom{r}{2} n^2 \binom{d}{2} (1/n)^2 < r^2 d^2 / 4$. So with probability at least $1/8$, (i)–(v) all hold and $b \leq r^2 d^2 < r^3 d^2 / 2$. Let $\{e_1, f_1\}, \dots, \{e_b, f_b\}$ be the butterflies. Beginning with $I = I' = \emptyset$, we construct I and I' in b steps. At the j th step, we add two disjoint edges $\{e_j, g_j\}$ of H to I , and add to I' two disjoint edges $\{e'_j, g'_j\}$, neither of which is in H , and satisfying $e_j \cup g_j = e'_j \cup g'_j$; this last property will ensure that $H - I + I'$ is d -regular. Property (v) and the choice of g_j will ensure the edges of I are independent, and hence so are the edges of I' .

To find these edges, consider $\{e_j, f_j\}$. Property (i) holds so let $e_j \cap f_j = \{u, v\}$: for convenience we assume $u \in V_1$ and $v \in V_2$. Let Q be the set of vertices in one of $e_1, f_1, \dots, e_b, f_b$ or in some edge of I or in some edge containing either u or v : then $|Q| \leq 2br + |I|r + 2dr \leq 4br + 2rd \leq 3r^3 d^2$. There are at most $|Q|d$ edges meeting Q , and at most $|Q|drd$ edges meeting these edges. But $|Q|drd \leq 3r^5 d^4 < nd$, and H has nd edges. Hence there is an edge g_j of H so that no edge of H meets both g_j and Q . Let x and y be the vertices of g_j in V_1 and V_2 respectively, and put

$e'_j = (e_j \setminus \{u\}) \cup \{x\}$, $g'_j = (g_j \setminus \{x\}) \cup \{u\}$. Since e'_j and g'_j meet Q , in v and u respectively, and both meet g_j , neither e'_j nor g'_j is in H .

By choice of Q and by (v), e_j and g_j are disjoint, $e_j \cup g_j = e'_j \cup g'_j$, and this set of $2r$ vertices is disjoint from any edge so far in I (and hence also disjoint from any edge in I'), and is also disjoint from any butterfly. Furthermore, adding e'_j to H does not create a butterfly: for if $\{e'_j, f\}$ is such a butterfly then f lies in H , $f \cap e_j \neq \emptyset$, $f \neq f_j$, so $|f \cap e_j| = 1$ and $x \in f$, contradicting the choice of g_j . Likewise $\{g'_j, h\}$ cannot be a butterfly, where h is in H , because $\{g_j, h\}$ is not a butterfly, implying $u \in h$ and $h \cap g_j \neq \emptyset$, another contradiction. So the addition of $\{e'_j, g'_j\}$ to H will not create a butterfly. Thus after b steps we reach sets I and I' , with $|I| = |I'| = 2b \leq r^3 d^2$, as described in the lemma. \square

Proof of Theorem 1.4. Take H satisfying Lemmas 6.1 and 6.2, and let $\hat{H} = H - I + I'$. By the properties of Lemma 6.2, \hat{H} is d -regular and simple. Let X be an independent set in \hat{H} . In H , X contains at most $|I| \leq r^3 d^2$ edges. Recalling that $n \geq r^5 d^4$, this means $H[X]$ has at most $n/2d^{1/(r-1)}$ edges and, since H satisfies property $I'(r, n, d)$, this means X satisfies (1). Therefore \hat{H} has property $I(r, n, d)$. Now suppose X is a set satisfying (2). Since H has property $D'(r, n, d)$, this means $H[X]$ is $(k-1)$ -degenerate, where $k = 4(\log \log d)/\log d$. But the edges of $\hat{H}[X]$ not in $H[X]$ are independent, so $\hat{H}[X]$ is k -degenerate. Therefore \hat{H} has property $D(r, n, d)$ also. \square

7. MORE ON PREFERENCE ORDERS

In this section we aim to establish some basic properties of $f(r, \theta)$ and $g(r, \alpha)$. However the notion of an (r, m) -preference order P and the definition of $f_P(\theta)$ are tailored to suit the proof of Theorem 1.5, and in themselves are somewhat cumbersome to work with. The value of $f_P(\theta)$ takes no account of any $x \in P$ with $x_i < \theta$ for some i , and for every $x \in P$ it takes no account of x_{i_x} , making some information in $\{x : x \in P\}$ appear redundant. Further, it can be difficult to manipulate simultaneously the r different orders in P .

These drawbacks are resolved by introducing the notion of a *cover*, which is nothing more than a perfect matching. Complete information about the function $f(r, \theta)$ can (in principle) be found by studying covers, without the complication and redundancy of preference orders. Moreover, to obtain a useful lower bound on $f(r, \theta)$ it is more or less necessary to work with covers.

7.1. Preference orders and covers.

Definition 7.1. For $r \geq 1$, an r -cover is an r -graph Q with $V(Q) \subset [0, 1]$ whose edges form a perfect matching: that is, $|V(Q)| = rn$ for some $n \in \mathbb{N}$ and the edge set $E(Q)$ of Q comprises n pairwise disjoint edges. We define

$$h(Q) = \max\{\prod_{y \in e} y : e \in E(Q)\}.$$

For $\theta \in [0, 1/(r+1))$, we define an (r, θ, n) -cover to be an r -cover Q with $V(Q) = \{\theta + (1/(r+1) - \theta)j/n : j \in [rn]\}$. We further define

$$h(r, \theta, n) = \min\{h(Q) : Q \text{ is an } (r, \theta, n)\text{-cover}\}$$

$$\text{and } h(r, \theta) = \inf\{h(r, \theta, n) : n \in \mathbb{N}\}.$$

Moreover we define $h(r, 1/(r+1)) = \lim_{\theta \rightarrow (1/(r+1))^-} h(r, \theta) = 1/(r+1)^r$.

Observe that $1/(r+1)$ is always in the vertex set of an (r, θ, n) -cover (when $j = n$); another way to represent the vertex set is in the form $\{1/(r+1) + jx : j = -n+1, -n+2, \dots, (r-1)n\}$ where $x = (1/(r+1) - \theta)/n$. Evidently $\theta^r <$

$h(r, \theta, n) \leq (\theta + r(1/(r+1) - \theta))^r$ for all n , so $\lim_{\theta \rightarrow (1/(r+1))^-} h(r, \theta) = 1/(r+1)^r$, as asserted in the definition.

Notice some differences between a cover and a preference order. The edges of Q are unordered subsets whereas $\{x : x \in P\}$ consists of ordered r -tuples. The value $h(Q)$ is the maximum, over *all* edges, of the product of *all* numbers that are vertices of the edge. We avoid numbers we are not interested in by specifying the vertex set of the cover: thus all the vertices (r, θ, n) -cover are larger than θ . Covers are easier to work with than preference orders, but the two are related.

Theorem 7.2. *Let $r \in \mathbb{N}$, $r \geq 2$ and let $\theta \in [0, 1/r]$. Then $f(r, \theta) = h(r-1, \theta)$.*

As might be expected, the proof of this theorem comes by somehow merging the r orders of P into one single cover, removing the redundant elements and performing small perturbations of the hypergraphs. In this context, we say that Q and Q' are *similar* if Q is an (r, θ, n) -cover and Q' is the unique (r, θ', n) -cover such that the bijection $\theta + (1/(r+1) - \theta)j/n \mapsto \theta' + (1/(r+1) - \theta')j/n$ between $V(Q)$ and $V(Q')$ takes edges of Q to edges of Q' .

Lemma 7.3. *Let Q be an (r, θ, n) -cover and Q' be an (r, θ', n) -cover. If Q and Q' are similar then $|h(Q) - h(Q')| \leq r2^r|\theta - \theta'|$.*

Proof. We may suppose that $\theta < \theta'$ and, putting $\delta = \theta' - \theta$, that $r2^r\delta < 1$ else the lemma is trivial. Let $\xi : V(Q) \rightarrow V(Q')$ be the bijection $\xi(\theta + (1/(r+1) - \theta)j/n) = \theta' + (1/(r+1) - \theta')j/n$. If $y = \theta + (1/(r+1) - \theta)j/n$, then $\xi(y) = y + \delta - j\delta/n$. Since $j \in [rn]$ we have $y - r\delta < \xi(y) \leq y + \delta$. If e is an edge of Q and e' is the corresponding edge of Q' then $\prod_{\xi(y) \in e'} \xi(y) \leq \prod_{y \in e} (y + \delta) \leq \prod_{y \in e} y + 2^r\delta$, so $h(Q) \leq h(Q') + 2^r\delta$. Likewise $\prod_{y \in e} y \leq \prod_{\xi(y) \in e'} (\xi(y) + r\delta) \leq \prod_{\xi(y) \in e'} \xi(y) + r2^r\delta$, so $h(Q) \leq h(Q') + r2^r\delta$. \square

To prove Theorem 7.2 we first bound h in terms of f .

Lemma 7.4. *Let $r \geq 2$, $\theta \in [0, 1/r]$ and $m \in \mathbb{N}$. Then $h(r-1, \theta, n) \leq f(r, \theta, m) + (r-1)2^{r-1}/m$ holds, where $n = m - r\lceil\theta m\rceil + r$.*

Proof. Take a preference order P on $[m]$ with $f_P(\theta) = f(r, \theta, m)$. Form an r -cover Q_1 with vertex set $\{i/rm : i \in [rm]\}$ by merging the r orders of P but reducing the values in the i th order by $(i-1)/rm$: that is, for each $x = (x_1, \dots, x_r) \in P$, Q_i has the edge $e(x) = \{x_1, x_2 - 1/rm, x_3 - 2/m, \dots, x_r - (r-1)/rm\}$. Observe that Q_1 is indeed an r -cover. Let $k = \lceil\theta m\rceil - 1$, so $k/m < \theta \leq (k+1)/m$. Then the condition $x_i \geq \theta$ for $1 \leq i \leq r$ is equivalent to $\min\{v : v \in e(x)\} > k/m$.

We shall transform Q_1 but keep the same vertex set. Let $A = \{1/rm, \dots, k/m\}$ be the rk smallest elements of $V(Q_1)$. For any r -cover Q with $V(Q) = V(Q_1)$, let $F(Q) = \{e \in E(Q) : e \cap A = \emptyset\}$. So $e(x) \in F(Q_1)$ if and only if $x_i > \theta$ for all i . For $e \in E(Q)$ let $\psi(e)$ be the product of the $(r-1)$ elements in e except the largest; then $\psi(e(x)) \leq \prod_{i \neq i_x} x_i$ for $e(x) \in E(Q_1)$. So, defining $\Psi(Q) = \max\{\psi(e) : e \in F(Q)\}$ we have $\Psi(Q_1) \leq f_P(\theta)$.

Let $B = \{1 - 1/r + 1/rm, \dots, 1\}$ be the m vertices greater than $1 - 1/r$. Suppose $e \cap B = \emptyset$ some edge e . Since $|B| = m = |E(G)|$ there must be some edge f with $|f \cap B| \geq 2$. Let u be the greatest element of e and v be the second greatest in f . Then $u \notin B$ and $v \in B$ so $u < v$. Form Q' from Q by replacing e and f by $e' = (e \setminus \{u\}) \cup \{v\}$ and $f' = (f \setminus \{v\}) \cup \{u\}$. Then $\psi(e') = \psi(e)$ and $\psi(f') \leq \psi(f)$, since $u < v$. Note that $F(Q') = F(Q)$, and so $\Psi(Q') \leq \Psi(Q)$. By repeating this operation as necessary we arrive at an r -cover Q_2 with $\Psi(Q_2) \leq f_P(\theta)$, $F(Q_2) = F(Q)$ and $|e \cap B| = 1$ for every edge $e \in E(Q_2)$.

Let $C = \{1 - 1/r - (r-2)k/m + 1/rm, \dots, 1 - 1/r\}$ be the $r(r-2)k$ vertices immediately below B . We show there is an r -order Q_3 with $\Psi(Q_3) \leq f_P(\theta)$,

$|e \cap B| = 1$ for every edge $e \in E(Q_3)$, $f \cap C = \emptyset$ for $f \in F(Q_3)$, and $f \subset A \cup B \cup C$ for every edge $f \notin F(Q_3)$. If either $k = 0$ or $r = 2$ we can take $Q_3 = Q_2$, in the first case because $A = C = \emptyset$ so $F(Q_2) = E(Q_2)$, and in the second case because $C = \emptyset$, and $|f \cap B| = |f \cap A| = 1$ for $f \notin F(Q_3)$. So we can assume $k > 0$ and $r > 2$; that is, $C \neq \emptyset$. Suppose that $|f \cap C| < r - 2$ for some edge $f \notin F(Q_2)$. Since $|C| = r(r - 2)k > 0$ and there are at most rk edges not in $F(Q_2)$ (because each contains a vertex of A), we have $e \cap C \neq \emptyset$ for some edge $e \in F(Q_2)$. Now $|f \cap B| = 1$; pick some $w \in f \cap A$, and then there exists $u \in f$, $u \notin B \cup C$ and $u \neq w$. Let $v \in e \cap C$; then $u < v$. Form Q'' from Q_2 by replacing e and f by $e'' = (e \setminus \{v\}) \cup \{u\}$ and $f'' = (f \setminus \{u\}) \cup \{v\}$. Notice $w \in f'' \cap A$ so $f'' \notin F(Q'')$, and $\psi(e'') \leq \psi(e)$ because $u < v$. Thus $\Psi(Q'') \leq \Psi(Q_2)$ and $|F(Q'')| < |F(Q_2)|$. Hence repeating this operation results in an r -order Q_3 with $\Psi(Q_3) \leq f_P(\theta)$, $|e \cap B| = 1$ for all $e \in E(Q_3)$ and $|f \cap C| = r - 2$ for every edge $f \notin F(Q_3)$. Thus $|f \cap A| = 1$ for all $f \notin F(Q_3)$. But $|C| = r(r - 2)k = (r - 2)|A|$ so C lies entirely within edges not in $F(Q_3)$; in other words, $|e \cap B| = 1$ and $e \cap C = \emptyset$ for every $e \in F(Q_3)$.

Let $V(Q_4) = V(Q) - A - B - C = \{k/m + 1/rm, \dots, 1 - 1/r - (r - 2)k/m\}$. Let the edges of Q_4 be the edges of $F(Q_3)$ with the element in B removed. By the properties of Q_3 , Q_4 is an $(r - 1)$ -cover. Note $|E(Q_4)| = |E(Q_3)| - |A| = m - rk = n$; so in fact, Q_4 is precisely an $(r - 1, k/m, n)$ -cover, because $V(Q_4) = \{k/m + jx : j \in [(r - 1)n]\}$ where $x = 1/rm = (1/r - k/m)/n$. By definition of $\Psi(Q_3)$ and of Q_4 we see that $h(Q_4) = \Psi(Q_3) \leq f_P(\theta)$.

Finally, let Q_5 be the $(r - 1, \theta, n)$ -cover that is similar to Q_4 . Since $k/m < \theta \leq k/m + 1/m$, Lemma 7.3 shows $h(Q_5) \leq h(Q_4) + (r - 1)2^{r-1}/m \leq f_P(\theta) + (r - 1)2^{r-1}/m$, and this proves the lemma. \square

Now we bound f in terms of h . The proof seeks to mimic, as far as possible, the reverse of the previous proof, though the steps are now much easier.

Lemma 7.5. *Let $r \geq 2$, $\theta \in [0, 1/r]$ and $n \in \mathbb{N}$. Then $f(r, \theta, rm) \leq h(r - 1, \theta, n) + (r - 1)2^{r-1}/m$ holds, where $m - r\lceil \theta m \rceil + r = n$.*

Proof. Take an $(r - 1, \theta, n)$ -cover Q with $h(Q) = h(r - 1, \theta, n)$. Choose m with $n = m - r\lceil \theta m \rceil + r$; such a choice is possible because the right hand side increases by at most one as m increases by one. Let Q_1 be the $(r - 1, k/m, n)$ -cover that is similar to Q , where $k = \lceil \theta m \rceil - 1$. By Lemma 7.3, $h(Q_1) \leq h(Q) + (r - 1)2^{r-1}/m$.

Now form an r -cover Q_2 with $V(Q_2) = \{1/rm, \dots, 1\} = V(Q_1) \cup A \cup B \cup C$, where $A = \{1/rm, \dots, k/m\}$, $B = \{1 - 1/r + 1/rm, \dots, 1\}$ and $C = \{1 - 1/r - (r - 2)k/m + 1/rm, \dots, 1 - 1/r\}$. For each edge e of Q_1 let $e \cup \{v\}$ be an edge of Q_2 , for some $v \in B$, and then add $m - n = rk$ further edges each comprising one vertex in A , one in B and $r - 2$ in C . Observe that it is possible to form an r -cover in this way, because $|V(Q_2)| = rm$, $E(Q_1) = n$, $|A| = rk$, $|B| = m$ and $|C| = r(r - 2)k$.

Finally, we form an (r, rm) -preference order P from Q_2 . For each edge $f = \{v_1, \dots, v_r\} \in E(Q_2)$, where $v_1 < \dots < v_r$, let each of the r -tuples $y_f^1, y_f^2, \dots, y_f^r$ belong to P , where $y_f^i = (v_{1+i}, v_{2+i}, \dots, v_{r+i})$, subscripts being evaluated modulo r . Note that for each $\ell \in [rm]$ there is a unique $x = (x_1, \dots, x_r) \in P$ with $x_i = \ell/rm$, and P is indeed an (r, rm) -preference order. Let $x \in P$ satisfy $\prod_{i \neq i_x} x_i = f_P(\theta)$. Then $x = y_f^i$ for some $f \in E(Q_2)$. Now $x_i \geq \theta$ for $1 \leq i \leq r$, so $u \geq \theta > k/m$ for all $u \in f$. Hence $f \cap A = \emptyset$, so $f = e \cup \{v\}$ for some $e \in E(Q_1)$ and some $v \in B$. Since $f \cap B = \{v\}$ we have $f(r, \theta, rm) \leq f_P(\theta) = \prod_{i \neq i_x} x_i = \prod_{z \in f, z \neq v} z = \prod_{z \in e} z \leq h(Q_1) \leq h(Q) + (r - 1)2^{r-1}/m$, proving the lemma. \square

When proving Theorem 7.2, we need consider only large m and n .

Lemma 7.6. *For $r \geq 2$, $0 \leq \theta < 1/r$ and $m, n, k \in \mathbb{N}$, $f(r, \theta, km) \leq f(r, \theta, m)$ and $h(r-1, \theta, kn) \leq h(r-1, \theta, n)$ hold. In particular, $f(r, \theta) = \liminf_{m \rightarrow \infty} f(r, \theta, m)$ and $h(r-1, \theta) = \liminf_{n \rightarrow \infty} h(r-1, \theta, n)$.*

Proof. Take an (r, m) -preference order P with $f_P(\theta) = f(r, \theta, m)$. Produce an (r, km) -preference order P' in the following natural way: if j is the number at relative position x in the i th order of P , then place $j, j+m, j+2m, \dots, j+(k-1)m$ at relative positions $x, x-1/km, x-2/km, \dots, x-(k-1)/km$ in the i th order of P' . Then if $x' \in [\theta, 1]^r$ and $x' \in P'$, there exists $x \in [\theta, 1]^r$ with $x \in P$ and $\prod_{i \neq i_{x'}} x'_i \leq \prod_{i \neq i_x} x_i$, and so $f(r, \theta, mk) \leq f_{P'}(\theta) \leq f_P(\theta) = f(r, \theta, m)$.

In like manner, if Q is an $(r-1, \theta, n)$ -cover with $h(Q) = h(r-1, \theta, n)$, then form an $(r-1, \theta, kn)$ -cover Q' thus. Note that $V(Q) \subset V(Q')$. For each $e \in E(Q)$ place the edges $e, e-1/rkn, \dots, e-(k-1)/rkn$ into $E(Q')$, where $e-y = \{x-y : x \in e\}$. It is easy to see that Q' is an $(r-1, \theta, kn)$ -cover and $h(Q') = h(Q)$. \square

Proof of Theorem 7.2. Let $\theta \in [0, 1/r)$. By Lemma 7.6 there is a sequence $(m_j)_{j=1}^\infty$ with $m_j \rightarrow \infty$ and $f(r, \theta, m_j) \rightarrow f(r, \theta)$. Let $n_j = m_j - r \lfloor \theta m_j \rfloor$. By Lemma 7.4, $h(r-1, \theta) \leq h(r-1, \theta, n_j) \leq f(r, \theta, m_j) + 2^{r-1}/m_j$ holds for all j , and taking the limit as $j \rightarrow \infty$ gives $h(r-1, \theta) \leq f(r, \theta)$. A corresponding argument, but using Lemma 7.5, shows that $f(r, \theta) \leq h(r-1, \theta)$, so $f(r, \theta) = h(r-1, \theta)$ for $\theta < 1/r$. When $\theta = 1/r$, we have $h(r-1, 1/r) = \lim_{\theta \rightarrow (1/r)^-} h(r-1, \theta) = (1/r)^{r-1}$ by definition. Thus, using the result for $\theta < 1/r$, we have $\lim_{\theta \rightarrow (1/r)^-} f(r, \theta) = (1/r)^{r-1}$. But we know (see after Definition 2.4) that $f(r, \theta)$ is decreasing and $f(r, 1/r) \geq (1/r)^{r-1}$. Therefore $f(r, 1/r) = (1/r)^{r-1} = h(r-1, 1/r)$, completing the proof. \square

7.2. Further properties. We now establish some basic properties of the functions $f(r, \theta)$ and $f(r, \theta, m)$, namely continuity, rate of convergence and initial constancy. In the light of Theorem 7.2 and Lemmas 7.4 and 7.5 we could derive these from corresponding properties of $h(r-1, \theta)$ and $h(r-1, \theta, n)$, and generally we do so since it is usually easier to argue in terms of covers than preference orders.

Lemma 7.7. *For $r \geq 1$ and $\theta, \theta' \in [0, 1/(r+1))$, $|h(r, \theta) - h(r, \theta')| \leq r2^r |\theta - \theta'|$. In particular, $h(r, \theta)$ is continuous for $\theta \in [0, 1/(r+1)]$.*

Proof. Let $\epsilon > 0$. Choose n so that $h(r, \theta, n) < h(r, \theta) + \epsilon$ and let Q be an (r, θ, n) -cover with $h(Q) = h(r, \theta, n)$. Let Q' be the similar (r, θ', n) -cover. By Lemma 7.3, $h(r, \theta') \leq h(Q') \leq h(Q) + r2^r |\theta - \theta'| \leq h(r, \theta) + r2^r |\theta - \theta'| + \epsilon$. So $h(r, \theta) - h(r, \theta') \leq r2^r |\theta - \theta'| + \epsilon$, and since this holds for all $\epsilon > 0$ we have $h(r, \theta) - h(r, \theta') \leq r2^r |\theta - \theta'|$. The same holds with θ and θ' interchanged, establishing the first half of the lemma, and hence also the continuity of $h(r, \theta)$ for $\theta \in [0, 1/(r+1))$. But $h(r, \theta)$ is continuous at $\theta = 1/(r+1)$ by definition of $h(r, 1/(r+1)) = \lim_{\theta \rightarrow (1/(r+1))^-} h(r, \theta)$. \square

The next lemma bounds how fast $f(r, \theta, m)$ converges to $f(r, \theta)$. Though we could derive this from a corresponding result for $h(r-1, \theta, n)$, we need only the bound on $f(r, \theta, m)$, and it is slightly quicker to prove this directly. The proof itself is straightforward: we choose a large preference order P' with $f_{P'}(\theta)$ close to $f(r, \theta)$, and from some randomly chosen elements $y \in P'$ we build an (r, m) -preference order P with $f_P(\theta)$ close to $f(r, \theta)$.

Lemma 7.8. *For $r \geq 2$, $0 \leq \theta \leq 1/r$ and $m \in \mathbb{N}$, $f(r, \theta) \leq f(r, \theta, m) \leq f(r, \theta) + 2^r \sqrt{(\log rm)/m}$ holds. In particular $f(r, \theta) = \lim_{m \rightarrow \infty} f(r, \theta, m)$.*

Proof. The lower bound holds by Definition 2.4. For the upper bound, let $\epsilon > 0$ and choose N with $f(r, \theta, N) \leq f(r, \theta) + \epsilon$. By Lemma 7.6 we may assume that N is as large as we wish, certainly larger than m . Let $P' = (\langle'_1, \dots, \langle'_r)$ be an

(r, N) -preference order with $f_{P'}(\theta) = f(r, \theta, N)$. Let $S = \{y \in P' : y \in [\theta, 1]^r\}$. By definition, $f_{P'}(\theta) = \max\{\prod_{i \neq i_y} y_i : y \in S\}$. Since, for each i , fewer than θN elements $y \in P'$ satisfy $y_i < \theta$, we have $|S| > (1 - r\theta)N$.

We now construct an (r, m) -preference order $P = (<_1, \dots, <_r)$. More precisely, we specify only $\{x : x \in P\}$, but this is enough to determine $f_P(\theta)$. Put $k = \lceil \theta m \rceil - 1$, so $k/m < \theta \leq (k+1)/m$. Let $q = m - rk$, so $q > m(1 - r\theta)$. Partition the relative positions into three sets $A = \{1/m, \dots, k/m\}$, $Q = \{(k+1)/m, \dots, 1 - (r-1)k/m\}$ and $B = \{1 - (r-1)k/m + 1/m, \dots, 1\}$, so $|A| = k$, $|B| = (r-1)k$ and $|Q| = q$. By definition, $f_P(\theta) = \max\{\prod_{i \neq i_x} x_i : x \in P, x \in (Q \cup B)^r\}$.

Begin by placing rk r -tuples x into P , so that each $x \in (A \cup B)^r$, and for each x there is a unique index j with $x_j \in A$ and $x_i \in B$ for $i \neq j$. It is possible to find such r -tuples because $|B| = (r-1)|A|$. We finish the construction of P by adding to P a further set R of q r -tuples (to be described), so that if $x \in R$ then $x \in Q^r$. Observe that, when this is done, $f_P(\theta) = \max\{\prod_{i \neq i_x} x_i : x \in R\}$ holds.

Note at this point that we may assume that $2^r \sqrt{(\log rm)/m} < 1$ and in particular $m \geq 4^r$, since otherwise the lemma is trivial because $f(r, \theta, m) \leq 1$. Likewise we may assume $q \geq 2r$, for otherwise, whatever the choice of R , $f(r, \theta, m) \leq f_P(\theta) = \max\{\prod_{i \neq i_x} x_i : x \in R\} \leq (1/r + q/m)^r \leq (1/r)^{r-1} + 2^{r-1}q/m \leq (1/r)^{r-1} + 2^r \sqrt{(\log rm)/m}$, using $m \geq 4^r$: the lemma then holds because $f(r, \theta) \geq (1/r)^{r-1}$ by Theorem 2.6. In particular $2r \leq q = m - rk \leq m - r\theta m + r$, so $1 - r\theta \geq r/m$. Since N is large we may therefore assume that $|S| > (1 - r\theta)N \geq rN/m \geq m \geq q$.

To find R , we turn to the large preference order P' , and choose a random subset $R' \subset S$ of size q (we know $|S| > q$). We then take R to be the q elements of Q^r whose relative orders are the same as those of R' . Formally, define an injection $\iota : R' \rightarrow Q^r$ so that if $y \in R'$ and $x = \iota(y)$ then $x_i = (k+j)/m$, where y_i is the j 'th largest element of $\{y'_i : y' \in R'\}$, meaning $j = |\{y' \in R' : y'_i \leq y_i\}|$. Then take $R = \iota(R')$. This completes the construction of P . What remains is to show there is a choice of R' such that $f_P(\theta)$ is suitably bounded.

We say $y \in S$ *spoils* $<_i$ if $y \in R'$ and $x_i > y_i + (r + \sqrt{2q \log rq})/m$, where $x = \iota(y) \in R$. What is the probability that y spoils $<_i$? Conditioned on the event $y \in R'$, the remaining $q-1$ elements of R' are chosen randomly from $S - \{y\}$. Let X be the subset of these taken from the subset $Y \subset S$ of elements whose i 'th co-ordinate exceeds y_i : that is, $Y = \{y' \in S : y'_i > y_i\}$ and $X = R' \cap Y$. Then $x_i = (k + q - |X|)/m$. Now $|X|$ is distributed hypergeometrically with parameters $|S| - 1, |Y|, q - 1$, with mean $\lambda = (q-1)|Y|/(|S|-1)$. Note that, by definition of S , there are at most $y_i N - \theta N$ elements of S not in Y , so $|Y| \geq |S| - y_i N + \theta N$.

Since $x_i = (k + q - |X|)/m$, we have $x_i = (m - (r-1)k - |X|)/m \leq 1 - (r-1)\theta + (r-1)/m - |X|/m$. Now $\lambda > (q-1)|Y|/|S| > q|Y|/|S| - 1$, because $|Y| < |S|$; thus $|Y| < |S|(\lambda + 1)/q$. So the inequality $|Y| \geq |S| - y_i N + \theta N$ means $y_i \geq |S|/N - |Y|/N + \theta > (|S|/N)(1 - (\lambda + 1)/q) + \theta$. Using $|S| > (1 - r\theta)N$ and $q > (1 - r\theta)m$ this gives $y_i > 1 - (r-1)\theta - (\lambda + 1)/m$. Therefore $x_i - y_i < (r - |X| + \lambda)/m$.

If y spoils $<_i$ then $x_i - y_i > (r + \sqrt{2q \log rq})/m$, and so $|X| < \lambda - \sqrt{2q \log rq}$. By Proposition 4.1, the probability of this is at most $e^{-(2q \log rq)/2\lambda} < e^{-\log rq} = 1/rq$. We say y *spoils* P if y spoils $<_i$ for some $i \in [r]$. Thus, conditional on $y \in R'$, the probability that y spoils P is less than $1/q$. The unconditional probability that $y \in R'$ is $q/|S|$, and so the expected number of elements $y \in S$ spoiling P is less than $|S|(q/|S|)(1/q) = 1$.

Hence there is some choice of R' for which no element spoils P , and $x_i - y_i \leq (r + \sqrt{2q \log rq})/m$ for every $y \in R'$ and every $i \in [r]$. We have $(r + \sqrt{2q \log rq})/m < r/m + \sqrt{(2/m) \log rm} < 2\sqrt{(\log rm)/m}$ because $m \geq 4^r$. There is some $x = \iota(y) \in R$ with $f_P(\theta) = \prod_{i \neq i_x} x_i$, and so $f(r, \theta, m) \leq f_P(\theta) = \prod_{i \neq i_x} x_i \leq \prod_{i \neq i_y} x_i \leq$

$\prod_{i \neq i_y} (y_i + 2\sqrt{(\log rm)/m}) \leq \prod_{i \neq i_y} y_i + 2^r \sqrt{(\log rm)/m}$. Since $y \in S$, so $\prod_{i \neq i_y} y_i \leq f_{P'}(\theta) \leq f(r, \theta) + \epsilon$, we have $f(r, \theta, m) \leq f(r, \theta) + \epsilon + 2^r \sqrt{(\log rm)/m}$. The bound holds for every $\epsilon > 0$, and so the lemma is proved. \square

We now explain why the function $f(r, \theta)$ is constant for small θ .

Definition 7.9. For each $r \geq 1$, let φ_r be the smallest solution to the equation $\theta(1 - 1/(r+1) - (r-1)\theta)^{r-1} = h(r, \theta) = f(r+1, \theta)$.

Note that there is a solution to this equation, because $h(r, 1/(r+1)) = (1/(r+1))^r$, and $h(r, \theta) = f(r+1, \theta)$ by Theorem 7.2. Moreover $h(r, \theta) \geq h(r, 1/(r+1)) > 0$ for all $\theta \in [0, 1/(r+1)]$, so $0 < \varphi_r \leq 1/(r+1)$.

Theorem 7.10. For each $r \geq 1$, $h(r, \theta) = f(r+1, \theta)$ is constant for $\theta \in [0, \varphi_r]$.

Proof. In a nutshell, we take an $h(r, \varphi_r, n)$ cover Q_0 with $h(Q_0) \approx h(r, \varphi_r)$ and then, given $\theta < \varphi_r$, we increase the vertex set $V(Q_0)$ above and below to obtain an $(r, \theta, n+\ell)$ -cover $V(Q_\ell)$ by adding edges containing the new vertices: the property of φ_r means that these new edges don't affect $h(Q_\ell)$, so $h(Q_\ell) = h(Q_0)$ and hence $h(r, \theta) \leq h(r, \varphi_r)$, which is what we are after. In practice the outline given needs to be perturbed a little, for technical reasons.

By the continuity of $h(r, \theta)$ (Lemma 7.7) and the definition of φ_r , we know that $\theta(1 - 1/(r+1) - (r-1)\theta)^{r-1} < h(r, \theta)$ for $\theta < \varphi_r$. Let $\epsilon > 0$. Since $h(r, \theta)$ is continuous we may choose $0 < \theta' < \varphi_r$ with $h(r, \theta') < h(r, \varphi_r) + \epsilon$. By properties of continuity there exists $\delta > 0$ such that $\theta(1 - 1/(r+1) - (r-1)\theta)^{r-1} < h(r, \theta) - \delta$ for $\theta \in [0, \theta']$. Because $\theta' < 1/(r+1)$ there is some (r, θ', n) -cover Q_0 with $h(Q_0) < h(r, \theta') + \epsilon$ where, by Lemma 7.6, n can be as large as we please. Then $V(Q_0) = \{1/(r+1) + jx : j = -n+1, -n+2, \dots, (r-1)n\}$ with $x = (1/(r+1) - \theta')/n$; we choose n so that $x < \delta$.

Let $\theta \in (0, \theta')$. Choose ℓ minimal so that $\theta' - \ell(1/(r+1) - \theta')/n \leq \theta$, and for $k = 0, 1, \dots, \ell$, define $\theta_k = \theta' - k(1/(r+1) - \theta')/n$. Thus $\theta_0 = \theta'$ and $\theta_\ell \leq \theta$. (Moreover, by increasing n again if necessary, we can guarantee that $\theta_\ell > 0$.) Observe that $(1/(r+1) - \theta_k)/(n+k) = (1/(r+1) - \theta')/n = x$. Hence if Q_k is an r -cover with $V(Q_k) = \{1/(r+1) + jx : j = -n-k+1, -n-k+2, \dots, (r-1)(n+k)\}$, then Q_k is an $(r, \theta_k, n+k)$ -cover, and $V(Q_0) \subset V(Q_1) \subset \dots \subset V(Q_\ell)$. We construct such covers by defining $E(Q_k) = E(Q_{k-1}) \cup \{e_k\}$, $k = 1, \dots, \ell$, where $e_k = \{1/(r+1) + jx : j = -n-k+1, (r-1)(n+k-1) + 1, \dots, (r-1)(n+k)\}$.

For each $k \geq 1$, $\prod_{y \in e_k} y < (\theta_k + x)(1 - 1/r - (r-1)\theta_k)^{r-1} \leq \theta_k(1 - 1/(r+1) - (r-1)\theta_k)^{r-1} + x < h(r, \theta_k)$ because $x < \delta$, and $h(r, \theta_k) \leq h(Q_k)$, because Q_k is an $(r, \theta_k, n+k)$ -cover. Therefore $h(Q_k) = \max\{\prod_{y \in e} y : e \in E(Q_k), e \neq y\} = h(Q_{k-1})$. Hence $h(r, \theta_\ell) \leq h(Q_\ell) = h(Q_0) < h(r, \theta') + \epsilon < h(r, \varphi_r) + 2\epsilon$. The outer inequality holds for all $\epsilon > 0$ so $h(r, \theta_\ell) \leq h(r, \varphi_r)$. But we know (comment after Definition 2.4) that $f(r, \theta)$ decreases with θ , meaning by Theorem 7.2 that $h(r, \theta)$ decreases, and so $h(r, \theta_\ell) = h(r, \varphi_r)$. Since $\theta_\ell \leq \theta < \varphi_r$ and h is decreasing, we have $h(r, \theta) = h(r, \varphi_r)$. \square

It is readily checked, say by taking logarithms and differentiating, that the function $\theta(1 - 1/(r+1) - (r-1)\theta)^{r-1}$ increases for $\theta \leq 1/(r^2-1)$ and decreases thereafter. For $r = 1$ the function is always increasing and because $h(1, \theta)$ is decreasing we have $\varphi_1 = 1/(r+1) = 1/2$. Likewise, for $r = 2$, the function is increasing for $\theta \in [0, 1/3] = [0, 1/(r+1)]$, and so $\varphi_2 = 1/(r+1) = 1/3$. Consequently Theorem 7.10 means both $h(1, \theta)$ and $h(2, \theta)$ are constant throughout, as are therefore $f(2, \theta)$ and $f(3, \theta)$ (though we knew this already for other reasons). To get information for other values of r we need a useful lower bound on $h(r, \theta)$, which is what we do next.

7.3. Lower bounds. A simple averaging argument provides an initial, but non-trivial, lower bound on $h(r, \theta) = f(r+1, \theta)$, in terms of the following function.

Definition 7.11. For $r \geq 1$ and $\theta \in [0, 1/(r+1))$, define

$$w(r, \theta) = e^{-r} \left(\frac{u(1 - 1/(r+1) - (r-1)\theta)}{u(\theta)} \right)^{1/(1/(r+1) - \theta)}$$

where $u(y) = y^y$ and $u(0) = 1$.

Lemma 7.12. Let $r \geq 1$ and $\theta \in [0, 1/(r+1))$. Then $f(r+1, \theta) = h(r, \theta) \geq w(r, \theta)$ holds. In particular, $h(r, 0) \geq (r/e(r+1))^r$.

Proof. Let Q be an (r, θ, n) -cover, with $V(Q) = \{\theta + jx : j \in [rn]\}$ and $x = (1/(r+1) - \theta)/n$. For $e \in E(Q)$ let $\pi(e) = \prod_{y \in e} y$. Then

$$h(Q) = \max_{e \in E(Q)} \pi(e) \geq \left(\prod_{e \in E(Q)} \pi(e) \right)^{1/n} = \left(\prod_{v \in V(Q)} v \right)^{1/n} = e^{-S}$$

where $nS = \sum_{v \in V(Q)} \log(1/v)$. Now $nxS \leq \int_{\theta}^{1-1/(r+1)-(r-1)\theta} \log(1/t) dt = -\log(u(1-1/(r+1)-(r-1)\theta) + \log(u(\theta)) + r(1/(r+1) - \theta))$. Hence $h(Q) \geq e^{-r} (u(1-1/(r+1)-(r-1)\theta)/u(\theta))^{1/(1/(r+1)-\theta)} = w(r, \theta)$ holds for every (r, θ, n) -cover Q , and, bearing in mind Theorem 7.2 and the definition of $h(r, \theta)$, this proves the lemma. \square

We explore the properties of $w(r, \theta)$ a little further. The next definition is close to that of φ_r in Definition 7.9.

Definition 7.13. For each $r \geq 1$, let ϕ_r be the smallest positive solution to the equation $\theta(1-1/(r+1)-(r-1)\theta)^{r-1} = w(r, \theta)$, where $w(r, \theta)$ is as in Definition 7.11.

Lemma 7.14. Let $r \geq 1$. Then the function $w(r, \theta)$ is increasing for $\theta \leq \phi_r$ and decreasing for $\theta \geq \phi_r$.

Proof. It is possible to prove the lemma by just calculating from the definitions, but it is more illuminating to interpret the result in terms of covers. We argue in a way parallel to the proof of Theorem 7.10; this time, to avoid excessive technicalities, we content ourselves with a detailed sketch.

Let n be very large and let Q_0 be the $(r, 0, n)$ -cover with $V(Q_0) = \{j/(r+1)n : j \in [rn]\}$ and edge set $\{e_k : 0 \leq k < n\}$ where e_0 comprises the least vertex and $(r-1)$ largest vertices, e_1 the second least and $(r-1)$ largest remaining vertices, and so on: that is, $e_k = \{(k+1)/(r+1)n, (rn-(r-1)k-r+2)/(r+1)n, \dots, (rn-(r-1)k)/(r+1)n\}$. Then Q_k , which is Q_0 with the edges and vertices of e_0, \dots, e_{k-1} removed, is an $(r, k/(r+1)n, n-k)$ -cover.

The product $\prod_{y \in e_k} y$ is very close to $p(\theta) = \theta(1-1/(r+1)-(r-1)\theta)^{r-1}$ (the more so as n grows). Recall from the comment at the end of §7.2 that $p(\theta)$ increases for $\theta \leq 1/(r^2-1)$ and decreases thereafter. It must therefore be that $\phi_r < 1/(r^2-1)$ in order for Definition 7.13 to be satisfied. In the proof of Lemma 7.12 we saw that $w(r, \theta)$ was very close to the r th power of the geometric mean of the vertices of the (r, θ, n) -cover Q . Hence if $\theta = k/(r+1)n$ for some k then $w(r, \theta)$ is very nearly the r th power of the geometric mean of $V(Q_k)$. If $\theta < \phi_r$ this quantity is greater than $p(\theta) \approx \prod_{y \in e_k} y$ so the mean of $V(Q_{k+1})$ is greater than that of $V(Q_k)$; thus $w(r, \theta)$ is increasing at this point.

On the other hand, when $p(\theta) > w(r, \theta)$ then $\prod_{y \in e_k} y$ exceeds the r th power of the geometric mean of $V(Q_k)$, so the mean of $V(Q_{k+1})$ will be less than that of $V(Q_k)$ and $w(r, \theta)$ will be decreasing. Certainly $p(\theta) > w(r, \theta)$ while $\phi_r < \theta < 1/(r^2-1)$ since $p(\theta)$ is increasing in this range and so $w(r, \theta)$ is perforce decreasing. But now, the fact that $p(\theta)$ decreases for $\theta \geq 1/(r^2-1)$ means that $\prod_{y \in e_k} y$ is a

decreasing function of k in the remaining range; that is, when moving from Q_k to Q_{k+1} we are always removing the edge with the largest product, so the mean of $V(Q_k)$ continues to decrease, and thus so does $w(r, \theta)$. \square

In the proof of his lemma it was seen that $p(\theta)$ increases for $\theta \leq \phi_r$. Comparing Definitions 7.9 and 7.13, and noting $h(r, \theta) \geq w(r, \theta)$ as stated in Lemma 7.12, we then observe that $\phi_r \leq \varphi_r$.

Definition 7.15. For $r \geq 1$ and $0 \leq \theta \leq 1/(r+1)$, let

$$H(r, \theta) = \begin{cases} w(r, \phi_r) & \text{for } \theta \leq \phi_r \\ w(r, \theta) & \text{for } \theta \geq \phi_r \end{cases}$$

where ϕ_r is as in Definition 7.13.

Lemma 7.14 means that $H(r, \theta)$ is a decreasing function of θ . The importance of $H(r, \theta)$ lies in the next result.

Theorem 7.16. Let $r \geq 1$ and $0 \leq \theta \leq 1/(r+1)$. Then

$$f(r+1, \theta) = h(r, \theta) \geq H(r, \theta).$$

Proof. Theorem 7.2 shows $f(r+1, \theta) = h(r, \theta)$. Lemma 7.12 shows $f(r+1, \theta) \geq w(r, \theta)$ for all θ , and it was noted after Definition 2.4 that $f(r+1, \theta)$ is decreasing. Thus, for $\theta \leq \phi_r$, $f(r+1, \theta) \geq f(r+1, \phi_r) \geq w(r, \phi_r) = H(r, \theta)$, and for $\theta \geq \phi_r$, $f(r+1, \theta) \geq w(r, \theta) = H(r, \theta)$. \square

As can be seen from the proofs of Theorem 7.10 and Lemma 7.12, what lies behind the bound in the theorem is this. If Q is an (r, θ, n) -cover, where n is large, and $\theta < \varphi_r$, then the edge product $\prod_{y \in e} y$ has no effect on $h(Q)$ if e contains an element less than φ_r . On the other hand, if $\theta > \varphi_r$, then $h(Q)$ is near to the lower bound $w(r, \theta)$ only if all edge products are more or less equal.

Surprisingly, it seems that such covers, where all edge products are roughly equal, might exist. The case of most immediate interest is $r = 3$. In this case, $\phi_3 = 0.070906\dots$ and $w(3, \phi_3) = 0.026227\dots$. Using a computer program to generate $(3, \phi_3, n)$ -covers, which aims to minimise the sum of edge products by switching pairs of edges in the manner of the proof of Lemma 7.4, we have examples of $(3, \phi_3, 10000)$ -covers Q with $h(Q) \leq 0.026232\dots$, meaning $h(3, \phi_3) \leq h(3, \phi_3, 10000) \leq 0.026232\dots$. Given that $\phi_3 \leq \varphi_3$ and that $h(3, \theta)$ is decreasing, this shows $h(3, \phi_3) \leq 0.026232\dots$ and so Theorem 7.10 implies $h(3, 0) \leq 0.026232\dots$. But by Theorem 7.16 we have $h(3, 0) \geq H(3, \phi_3) = w(3, \phi_3) = 0.026227\dots$. In summary, $0.026227\dots \leq h(3, 0) = f(4, 0) \leq 0.026232\dots$.

Having tried the computer program on a few other pairs (r, θ) , we are led to make the following conjecture.

Conjecture 7.17. Equality holds in Theorem 7.16 for all r and θ .

For what it's worth, we remark that, if true, this conjecture would imply $\varphi_r = \phi_r$.

7.4. Proofs of Theorems 1.6 and 2.6. We have already proved most of the properties of $f(r, \theta)$ stated in Theorem 2.6; to finish the proof, and to derive Theorem 1.6 about $g(r, \alpha)$, we need only add a few more observations.

Proof of Theorem 2.6. We noted after Definition 2.4 that $f(r, \theta)$ is decreasing, and Lemma 7.7 (together with Theorem 7.2) shows $f(r, \theta)$ is continuous, giving assertion (a) of the theorem. Assertion (b) was established as part of the proof of Theorem 7.2. As for (c), let $P = (\prec_1, \dots, \prec_r)$ be an (r, m) -preference order and let P' be the $(r-1, m)$ -preference order $(\prec_1, \dots, \prec_{r-1})$. If $x = (x_1, \dots, x_r) \in P$ and $\min x_i \geq \theta$ then $x' = (x_1, \dots, x_{r-1}) \in P'$, and $\prod_{i \neq i_x} x_i \leq \prod_{i \neq i_{x'}, i \neq r} x_i$, so

$f_P(\theta) \leq f_{P'}(\theta)$, which implies (c). Assertion (d) was noted already after Definition 2.4, and again after Theorem 7.10.

By Definition 7.9, Theorem 7.10 and Lemma 7.12, we have $\varphi_{r-1}(1 - 1/r)^{r-2} \geq \varphi_{r-1}(1 - 1/r - (r-2)\varphi_{r-1})^{r-2} = h(r-1, \varphi_{r-1}) = h(r-1, 0) \geq ((r-1)/e)^{r-1}$, so $\varphi_{r-1} \geq (1 - 1/r)e^{-r+1}$. In the light of Theorem 7.10, assertion (e) follows. Assertion (f) was explained immediately before Conjecture 7.17. The first inequality of assertion (g) is part of Lemma 7.12, given that $f(r, 0) = h(r-1, 0)$. For the second inequality, consider the $(r-1, 0, n)$ -cover Q with $V(Q) = \{j/rn : j \in [(r-1)n]\}$ and edge set $E(Q) = \{e_k : k \in [n]\}$ where $e_k = \{k/rn, (k+n)/rn, (k+(r-2)n)/rn\}$. Then $h(Q) = \prod_{y \in e_n} y = (r-1)!/r^{r-1}$, and $f(r, 0) = h(r-1, 0) \leq h(Q)$. This completes the proof. \square

Proof of Theorem 1.6. We appeal throughout to the properties of $f(r, \theta)$ given in Theorem 2.6 and to the fact that $g(r, \alpha) = -1/\log_r f(r, \beta(\alpha))$.

By Definition 2.5, $\beta(\alpha)^\alpha = f(r, \beta(\alpha))$ so, by Theorem 2.6 (a), $\beta(\alpha)$ is continuous in α and, by the remark following the definition, strictly increasing. Again appealing to Theorem 2.6 (a) we see that $g(r, \alpha)$ is continuous and decreasing, which is assertion (a). Assertion (b) is a consequence of Theorem 2.6 (b) and assertion (c) follows from Theorem 2.6 (d).

Let $r \geq 4$. Define δ by $\varphi_{r-1}^{1+\delta} = f(r, \varphi_{r-1})$. By Definition 2.5, $\beta(1+\delta) = \varphi_{r-1}$; by Theorem 7.10 and the fact that $\beta(\alpha)$ is increasing, $g(r, \alpha)$ is constant for $\alpha \leq 1+\delta$, so to prove assertion (d) it is enough to show that $\delta > 1/(r+3)$. In the previous proof we showed that $\varphi_{r-1}(1 - 1/r)^{r-2} \geq h(r-1, \varphi_{r-1}) = f(r, \varphi_{r-1})$, so $\varphi_{r-1}^\delta \leq (1 - 1/r)^{r-2}$. We also showed $\varphi_{r-1} \geq (1 - 1/r)e^{-r+1}$. Hence $[(1 - 1/r)e^{-r+1}]^\delta \leq (1 - 1/r)^{r-2}$, and so $e^{-\delta(r-1)} \leq (1 - 1/r)^{r-2-\delta} < e^{(r-2-\delta)/r}$, or $-\delta(r-1) < (r-2-\delta)/r$. Thus $\delta > (r-2)/(r^2 - r + 1)$, and, since $r \geq 4$, this implies $\delta > 1/(r+3)$ as desired.

Assertion (e) follows from the bounds $0.026227 \leq f(4, 0) \leq 0.026233$ mentioned before Conjecture 7.17, and (f) follows straightaway from Theorem 2.6 (g), which completes the proof. \square

8. PROPERTY B

An ℓ -uniform hypergraph H is k -colourable if its vertices can be coloured with k colours so that no edge is monochromatic, and $\chi(H)$ is the smallest k for which H is k -colourable. Erdős [7, 8] studied the minimum number of edges in a bipartite hypergraph H — that is, $\chi(H) = 2$: such hypergraphs are said to have “Property B”.

Let $m(\ell, r)$ be the minimum number of edges in an ℓ -graph H with $\chi(H) > r$. Let $Q(r, \ell)$ be the minimum number of vertices in an r -partite r -graph G with list chromatic number $\chi_\ell(G) \geq \ell$. Extending the result of Erdős, Rubin and Taylor [9], who proved the case $r = 2$, Kostochka [16] proved that $m(\ell, r)$ and $Q(r, \ell)$ are closely tied: indeed $m(\ell, r) \leq Q(r, \ell) \leq rm(\ell, r)$.

There has been no significant improvement on the upper bound for $m(\ell, 2)$ since Erdős [8] proved $m(\ell, 2) \leq \ell^2 2^\ell$. The lower bound has been improved a few times, the best to date being $m(\ell, 2) = \Omega((\ell/\log \ell)^{1/2} 2^\ell)$ by Radhakrishnan and Srinivasan [18]. A simple proof of this bound, and of the generalisation $m(\ell, r) = \Omega((\ell/\log \ell)^{1-1/r} r^\ell)$, was given by Cherkashin and Kozik [6].

The method of [6] is close to that of Pluhár [17]. If an ℓ -graph H has fewer edges than is stated in the bound, then a random argument shows there is some ordering of the vertices without any chain of edges e_1, e_2, \dots, e_r , such that the last vertex of e_i is the first of e_{i+1} , $1 \leq i \leq r-1$. A simple greedy colouring algorithm then colours H with r colours.

The relevant part of the proof in [9] and [16] that relates $m(\ell, r)$ to $Q(r, \ell)$ is as follows: let G be a complete r -partite r -graph with $|E(H)|$ vertices in each class. Consider $V(H)$ to be a palette and let $E(H)$ be assigned as lists to each vertex in V_i , $1 \leq i \leq r$. If G can be coloured from these lists then $\chi(H) \leq r$. Any list colouring algorithm can thus be translated to give some lower bound on $m(\ell, r)$.

Our colouring algorithm for complete r -partite r -graphs selects some preference order P , after which each vertex $v \in V_i$ chooses the colour in $L(v)$ most preferred by $<_i$. In the case $r = 2$, where we choose $<_1$ to be the identity and $<_2$ to be its reverse, the translation is to find an ordering of the vertices of H without a chain e_1, e_2 and then to colour the first vertex of each edge red and the last blue. This is not quite the same as the method of [6] but effectively equivalent, and the bound obtained on $m(\ell, 2)$ is the same.

However our method makes no use of the fact that the lists in each V_i are the same, and for $r > 2$ the translated method is less effective than the method in [6], though it does show $m(\ell, r) = \Omega((\ell/\log \ell)^{1/2} r^\ell)$.

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